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# (Multi)matrix models and interacting clones of Liouville gravity

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Abstract: Large-$N$ matrix models coupled via multitrace operators are used to define, via appropriate double-scaling limits, solvable models of interacting multi-string theories. It is shown that although such theories are non-local at the world-sheet level they have a simple description of the spacetime physics. Such theories share the main characteristics of similarly coupled higher-dimensional CFTs. An interpretation has been given in the past of similar continuum limits in terms of Liouville interactions that violate the Seiberg bound. We provide a novel interpretation of this relation which agrees with the current understanding of Liouville theory and analogous observations in the AdS/CFT correspondence.

Keywords: Matrix Models, Gauge-gravity correspondence, Bosonic Strings.

## Contents

# 1. Introduction

## 1.1 Setup and questions

In the context of the AdS/CFT correspondence, it has been argued [1, 2] that the holographic dual of the product of $k$ conformal field theories (CFTs) in $d$ dimensions, deformed by a set of multi-trace interactions that couple the CFTs together, is a theory of quantum (multi)-gravity (or better a multi-string theory) on a union of $k$ $AdS_{d+1}$ spaces with a formally common boundary — the boundary being isomorphic to the space where the dual product CFT lives. Assuming that each CFT has an independent gauge group $G_i$ and that there are no fields charged under more than one group, multi-trace interactions are the only way to couple the CFTs. We will assume in this paper that we are working in the large-$N$ limit.

From the quantum gravity point of view this is an interesting setup for the following reasons:

($i$) This is a non-trivial example of an interacting multi-graviton theory with a UV completion (the completion provided by the dual CFT). $1/N$ effects generate a non-vanishing potential for the gravitons making some of them massive. This effect appears as follows: on the field theory side, the multi-trace interactions violate the conservation of the energy-momentum tensor of each CFT, they retain, however, the conservation of a total stress tensor. On the dual gravity side, a linear combination of the $k$ original gravitons remains massless while the rest of the gravitons obtain non-zero masses of order $1/N$. One can think of this effect as a Higgs mechanism for gravity [3].

Coupling more than one gravitons together, or giving the gravitons a mass, have been long standing theoretical problems [4 − 10] (see [11] for a recent review). The issue at stake is the possibility of coupling non-trivially massless gravitons, or giving them a mass without rending the theory UV sensitive. So far there are no-go theorems for non-trivial couplings among massless gravitons [10]. On the other hand, most attempts to write an effective low-energy action for massive graviton theories are plagued by serious problems — ghosts, instabilities and strong coupling problems.

This behavior is not only characteristic of theories where a graviton has a mass term in the Lagrangian, but also more exotic cases where the massive graviton can be a resonance. This happens when gravity is induced on branes, with the most celebrated example being given by the DGP model [12]. Indeed it was observed in [13, 14] that this theory exhibits a similar non-decoupling behavior as the standard Pauli-Fierz theory. It was subsequently shown that the theory becomes strongly coupled at hierarchically low scales [15, 16]. Moreover, as has already been analyzed in detail in [5, 8] massive graviton theories are generically unstable. Instabilities similar to those of the Pauli-Fierz theory also occur in brane-induced gravity as reviewed in [17].

In this respect, having an example with a UV complete, non-perturbative formulation, where these issues can be addressed, is important not only from an academic desire to know if such theories exist, but also for potential phenomenological applications. Infrared modifications of gravity of this type could be useful, for example, in the resolution of the cosmological constant.

(*ii*) It is interesting to ask if and how standard properties of string theory and gravity (at the classical and quantum level) are modified in a multi-verse of interacting worlds. One can imagine new qualitative features on the string theory worldsheet and new dynamics in spacetime with potentially useful applications — for instance, potential applications in cosmology (*c.f.* [18] for a search of cosmological solutions in a multi-gravity theory).

The above AdS/CFT example offers a concrete arena to study the possibility and structure of such effects. In the context of the AdS/CFT correspondence, it has been argued [19] that multi-trace deformations on the boundary theory re-arrange the string perturbation theory in the bulk leading to a non-local string theory (NLST) with non-local dynamics on the worldsheet. One would like to know the precise rules of such dynamics. In particular, we would like to know whether such theories really define a new universality class of string theories. In this paper, we will have the chance to address some of these questions in a set of low-dimensional examples, where string theory is solvable.

### 1.2 Deformed matrix model products and non-critical NLSTs

The example of product CFTs and string theory on a union of AdS spacetimes can be generalized if we consider the product of more general quantum field theories (QFTs). As long as each of these QFTs has a string theory dual, the dual of the product theory will be a string theory on a union of spacetimes.

One of the first and best understood examples of holographic duality in string theory is the duality between the double scaling limit of large $N$ matrix models and two-dimensional quantum gravity coupled to conformal matter with central charge $c \leq 1$, *i.e.* $c \leq 1$ non-critical strings (for a review see [20, 21] and references therein). This is a good context for some of the above questions, because string theory in these examples is solvable at all orders in perturbation theory. Unfortunately, this case will not allow us to address the issues raised in point (*i*) above – the spacetime theory is a theory of scalar fields, hence we cannot arrange for a multi-gravity theory in this context. It allows us, however, to address some of the questions in point (*ii*) in a precise manner. Higher dimensional AdS/CFT examples that involve a multi-gravity theory and the related issues in point (*i*) will be discussed separately in a companion paper [22] (see, however, subsection 1.3 below and the discussion in section 5 for a summary of the main results).

The precise setup we want to consider in this paper is as follows. The "boundary" theory involves a product of $k$ large-$N$ matrix models $\prod_{i=1}^{k} \mathcal{M}_i$. The simultaneous standard double-scaling limit in each factor $\mathcal{M}_i$ gives the holographic description of a product of $c \leq 1$ non-critical string theories $\prod_{i=1}^{k} \mathcal{S}_i$, where each factor in this product is independent and does not communicate with the rest. We want to deform the product $\prod_{i=1}^{k} \mathcal{M}_i$ by adding multi-trace interaction terms to the total matrix model Lagrangian that couple different $\mathcal{M}_i$'s together. The non-trivial question is: under what circumstances can we find new double-scaling limits with non-vanishing multi-trace interaction couplings? Such limits would define holographically a multi-verse of interacting $c \leq 1$ non-critical string theories.

In section 2 we analyze the product of two Hermitian one-matrix models deformed by a general double-trace deformation. We will find that in this case a one-parameter family of double scaling limits exists with a non-trivial coupling between the two matrix models provided that the scaling properties of the single-trace operators that participate in the deformation are the same and that we tune the double-trace parameters to a special set of values. The free energy $\tilde{F}$ of the deformed theory is no longer the direct sum of the free energies $F_1$ and $F_2$ of matrix models $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. It is the logarithm of a bilateral Laplace transform

$$\tilde{F}(\tilde{t}_+, t_-) = \log \int_{-\infty}^{\infty} dt_+ \; e^{\tilde{t}_+ t_+ + F_1(\sin\theta \; t_+ + \cos\theta \; t_-) + F_2(\cos\theta \; t_+ - \sin\theta \; t_-)} , \tag{1.1}$$

where $\theta \in [0, 2\pi)$ is a free angular variable parametrizing a U(1) subspace of the three-dimensional double-trace coupling space and $\tilde{t}_+, t_-$ are scaling parameters in the modified two-matrix model. This result is a natural generalization of a similar formula that arises in double-trace deformations of single Hermitian one-matrix models [23, 24]. By holography, it gives the free energy of two *coupled* minimal string theories.

In section 3 we find that the genus expansion of correlation functions in the deformed theory (1.1) boils down, when expressed in terms of correlation functions in the undeformed product $\mathcal{M}_1 \otimes \mathcal{M}_2$, to a diagrammatic expansion where one has to sum over a series of terms that involve surfaces with contact interactions between the world-sheets of theory 1 and 2. This makes the world-sheet theory non-local. At tree-level, where (1.1) reduces to a Legendre transform, all the contributing world-sheets are genus zero, however, at any higher loop order $g \geq 1$ there is a series of contributions from touching surfaces with different geni less or equal than $g$. This is a concrete example of a non-local string theory expansion along the lines of [19].

In higher dimensional AdS/CFT systems, in a limit where string theory in the bulk reduces to semi-classical gravity (this is the limit of large 't Hooft coupling for a gauge theory on the boundary), there is, at tree-level, an alternative reformulation of the gravity dual of multi-trace deformations as ordinary gravity with mixed boundary conditions for the dual fields [25 – 29].

A similar reformulation of the tree-level theory appears to be possible in the context of matrix models and non-critical strings. This is in effect an old observation of Klebanov [23], who pointed out that the string susceptibility exponents and tree-level correlation functions in the double-trace deformed theory can be interpreted in the dual string theory as a change of the Liouville dressing of the tachyon condensate from the right branch to the wrong branch. At face value, this is a peculiar statement in Liouville theory. It would seem to suggest that we have to change Liouville theory in a drastic way by modifying the boundary conditions of the tachyon field at infinity. This is at odds with our current understanding of Liouville theory, where both branches (or equivalently both the standard and the dual cosmological constant interactions) have to be present in order to account for the right structure of correlation functions [30 − 34]. To the same effect, the only solution of the Wheeler-DeWitt equation that is regular in the strong coupling region has weak coupling asymptotics with both branches turned on [21]. From this point of view, we can change the weak coupling asymptotics at the expense of introducing a singularity in the strong coupling region. The singularity could have a physical origin, *e.g.* it could be attributed to a bunch of ZZ branes [35] localized in the strong coupling region. It is unclear, however, why ZZ branes would suddenly make an appearance in this context.

In section 4 we will propose a different interpretation of the observations in [23], which agrees with the current understanding of Liouville theory and analogous observations in the AdS/CFT correspondence. The basic point is this. In Liouville theory the cosmological constant $\mu$ and the dual cosmological constant $\tilde{\mu}$ are both present and have a fixed relation. The theory is not modified if we exchange at the same time $\mu \leftrightarrow \tilde{\mu}$ and the branch of the Liouville interaction. However, each of these transformations separately will give a different theory. We propose that this theory is what the modified matrix model describes at tree-level. In this modification Liouville theory continues to obey the usual rules. What we modify is the definition of the scaling parameter or in other words the specifics of the bulk/boundary dictionary. This is precisely what happens also in the AdS/CFT correspondence when we talk about mixed boundary conditions.

Beyond tree-level the standard free energy evaluated as a function of $\tilde{\mu}$ does not agree with the exact results obtained from the Laplace transform (1.1). This implies that the modified theory in the bulk is truly in a new universality class of string theories with a non-local worldsheet theory as envisioned in [19]. These points will be discussed further in section 4 (and appendix C).

There are several extensions of the modified product of two Hermitian one-matrix models that we explore in the main text (section 2 and appendices A, B). One is the extension to matrix quantum mechanics and the dual bosonic $c = 1$ string theory. Another is the possibility to couple more than two non-critical string theories either by interactions that couple pairs of string theories or by higher-order 'vertices' that involve higher multi-trace interactions. As an illustration, appendix B considers double scaling limits in a product of three Hermitian one-matrix models coupled by a triple-trace interaction.

### 1.3 AdS/CFT and multi-gravity

A direct analog of a double-scaled matrix model in higher dimensions is a conformal field theory. In generic products of CFTs deformed by multi-trace interactions, the multi-trace couplings break conformal invariance. If $\Phi = \prod_{i=1}^{n} \mathcal{O}_i$ is the perturbing operator in terms of a set of single-trace operators $\mathcal{O}_i$, where $i$ labels the $i$-th CFT, then there is an upper limit on $n$ in order for $\Phi$ to be perturbatively relevant or marginal. This upper limit depends crucially on the spacetime dimension $d$ through the unitarity bound. For scalar operators $\mathcal{O}_i$ the upper limit is 2 for $d \geq 6$, 3 for $d = 5, 4$, 5 for $d = 3$ and infinity for $d = 2$.

For concreteness, let us consider the case of two CFTs deformed by a double-trace interaction. Analyzing the one-loop $\beta$-functions of single-trace and multi-trace couplings in conformal perturbation theory one recovers the following picture [22].

At tree-level, *i.e.* to leading order in the $1/N$ expansion, there is a one-parameter family of non-trivial fixed points provided that the single-trace operators $\mathcal{O}_i$ participating in the double-trace deformation have the same scaling dimensions (or scaling dimensions with a sum equal to the spacetime dimension). This is one of the features that we encounter also in the corresponding matrix model analysis. In addition, one finds that the fixed points are repellors of the renormalization group (RG) equations, and perturbing away from them either drives the theory towards a fixed point, where the coupling between the CFTs vanishes, or towards strong coupling outside the range of validity of conformal perturbation theory.

The dual description of this system involves quantum gravity on the union of two AdS spaces with mixed boundary conditions for the scalar fields that are dual to $\mathcal{O}_i$ [1, 2]. At tree-level, there are two massless, non-interacting gravitons in this system and the RG running of the double-trace couplings on the boundary is encoded subtly in the mixed boundary conditions (it is not visible, in particular, as a radial running of the background solution away from AdS).

$1/N$ effects modify this picture in an interesting way. On the gauge theory side, $1/N$ effects shift the submanifold of fixed points and induce an RG running of single-trace operators. On the gravity side, a non-trivial potential is generated for the gravitons leading to an interacting bi-gravity theory in the general spirit of Kogan and Damour [36]. A linear combination of the gravitons remains massless, the orthogonal one obtains a mass of order $1/N$ [1, 2]. The quantum generated potential backreacts to the original AdS solution and for generic bare values of the double-trace couplings the bulk solution is no longer the union of two AdS spaces. It is rather a union of two spaces that are radially deformed and interpolate between AdS spaces.

In cases, where we can trust the boundary conformal perturbation theory, the gauge theory analysis predicts that the IR geometries are a union of AdS spaces with trivial coupling, hence in this region of spacetime there are again two massless, non-interacting gravitons. In a sense, the theory is dynamically washing away the mass of the graviton. The only way to retain a non-trivial bi-gravity theory everywhere in spacetime is to fine-tune the bare values of the double-trace couplings on a special one-dimensional submanifold of fixed points. The general lesson seems to be that massive gravity theories can in principle exist in a non-perturbative, well-defined quantum gravity context, but are not generic and require a high degree of fine-tuning. For a more detailed discussion of the general AdS/CFT case we refer the reader to [22].

In the rest of this paper, we will deal with a solvable toy model of this general setup that involves matrix models and non-critical strings. The analogies between the matrix model and AdS/CFT setups will be summarized in the concluding section 5.

## 2. Matrix models and a multiverse of $c \leq 1$ string theories

Multi-trace deformations in the context of matrix models and their implications for Liouville gravity were discussed originally in [37, 23, 24]. In this section, we will consider the effect of multi-trace deformations in a product of matrix models. Extending the analysis of [24], we are looking for new double scaling limits that define holographically an interacting union of $c \leq 1$ non-critical string theories. As a first concrete illustration, we will discuss the case of two large-$N$ Hermitian matrix models coupled by a double-trace deformation. Possible generalizations will be discussed in the last subsection 2.3 and appendix B.

### 2.1 Hermitian matrix models and minimal string theories

Let us begin by recalling some basic facts about multi-critical Hermitian one-matrix models and their double scaling limits. The partition function of the $k$-th multicritical one-matrix model is defined by the matrix integral

$$\mathcal{Z}_k = \int d\Phi \; e^{-N\left[\mathrm{Tr}\, V_k(\Phi) + (c_2+\lambda)\,\mathrm{Tr}\,\Phi^4\right]} , \tag{2.1}$$

with

$$V_k(\Phi) = \sum_{i=1}^{k} (-)^{i+1} c_i \Phi^{2i} \tag{2.2}$$

and $c_i$ a set of known constants [38]. In (2.1) we have chosen to consider a deformation of the potential by a term proportional to the single-trace operator $\mathrm{Tr}\,\Phi^4$. This model has a double scaling limit, where $N \to \infty$ and $c_2 + \lambda \to 0$ with the scaling variable $t \sim (c_2+\lambda) N^{2k/(2k+1)}$ kept fixed. In this limit, the partition function $\mathcal{F}_k = \log \mathcal{Z}_k$ becomes a function of the scaling variable $t$ and the matrix model provides the holographic formulation of the minimal $(2, 2k-1)$ bosonic string [38].

We will consider a two-matrix model, which arises from the direct product of a $k$-th and a $p$-th multi-critical matrix model $\mathcal{M}_{2,2k-1} \otimes \mathcal{M}_{2,2p-1}$ by a double-trace deformation. The standard double scaling limit of this theory at zero double-trace coupling describes the decoupled union of a $(2, 2k-1)$ and a $(2, 2p-1)$ minimal string.

#### 2.1.1 A prototype: deforming the $\mathcal{M}_{2,3} \otimes \mathcal{M}_{2,3}$ product

The theory we want to solve is a double-trace deformation of the $\mathcal{M}_{2,3} \otimes \mathcal{M}_{2,3}$ two-matrix model with partition function

$$\mathcal{Z} = \int D\Phi_1 D\Phi_2 e^{-N_1\,\mathrm{Tr}\left[\frac{1}{2}\Phi_1^2 + \lambda_1 \Phi_1^4\right] - N_2\left[\frac{1}{2}\Phi_2^2 + \lambda_2 \Phi_2^4\right] - \left[g_{11}(\mathrm{Tr}\,\Phi_1^4)^2 + g_{22}(\mathrm{Tr}\,\Phi_2^4)^2 + 2g_{12}\,\mathrm{Tr}\,\Phi_1^4\,\mathrm{Tr}\,\Phi_2^4\right]} . \tag{2.3}$$

The double-trace deformation involves the operators $\mathrm{Tr}\,\Phi_1^4$, $\mathrm{Tr}\,\Phi_2^4$. It is $g_{12}$ that couples the two separate matrix models, but $g_{11}$ and $g_{22}$ will play an important rôle when we take double scaling limits. The ranks of the matrices $\Phi_1, \Phi_2$ are $N_1, N_2$ respectively. In this paper we use the notation

$$N_1 \equiv N\,, \quad N_2 \equiv \nu N\,. \tag{2.4}$$

In the large $N$ limit both $N_1$ and $N_2$ will be scaled to infinity; the ratio $\nu$ will be kept fixed and will be treated as an extra parameter of the system. The single-trace and double-trace coupling constants are such that the overall Lagrangian scales like $N^2$.

It will be convenient to re-express the double-trace deformation as a sum of two squares

$$\begin{aligned} &g_{11}(\mathrm{Tr}\,\Phi_1^4)^2 + g_{22}(\mathrm{Tr}\,\Phi_2^4)^2 + 2g_{12}\,\mathrm{Tr}\,\Phi_1^4\,\mathrm{Tr}\,\Phi_2^4 = \\ &\qquad = r_1\left(\cos\theta\,\mathrm{Tr}\,\Phi_1^4 + \sin\theta\,\mathrm{Tr}\,\Phi_2^4\right)^2 + r_2(-\sin\theta\,\mathrm{Tr}\,\Phi_1^4 + \cos\theta\,\mathrm{Tr}\,\Phi_2^4)^2\ . \end{aligned} \tag{2.5}$$

Setting for quick reference

$$C \equiv \cos\theta\,, \quad S \equiv \sin\theta \tag{2.6}$$

we deduce the double-trace coupling relations

$$g_{11} = r_1 C^2 + r_2 S^2\,, \quad g_{22} = r_1 S^2 + r_2 C^2\,, \quad g_{12} = (r_1 - r_2) SC\ . \tag{2.7}$$

With these definitions

$$\det g = g_{11}g_{22} - g_{12}^2 = r_1 r_2\,, \quad \mathrm{Tr}\, g = g_{11} + g_{22} = r_1 + r_2\ . \tag{2.8}$$

We shall distinguish between two different cases: $(i)$ $r_1 r_2 \neq 0$ and $(ii)$ $r_1 r_2 = 0$. In the first case, the double-trace deformation consists of two quadratic terms. In the second case, there is a single quadratic term or no deformation at all.

**Case $(i)$: $r_1 r_2 \neq 0$.** Let us define the free energy $\hat{\mathcal{F}}$ of the single-trace theory as

$$e^{\hat{\mathcal{F}}[\lambda, N^2]} = \int D\Phi\ e^{-N\,\mathrm{Tr}\left[\frac{1}{2}\Phi^2 + \lambda\Phi^4\right]}\ . \tag{2.9}$$

A common trick to deal with the double-trace deformations is to use the identity

$$e^{g\mathcal{O}^2} = \frac{N}{\sqrt{4\pi g}} \int_{-\infty}^{\infty} dy\ e^{-\frac{N^2 y^2}{4g}}\, e^{Ny\mathcal{O}}\ . \tag{2.10}$$

In the case of (2.3), this trick allows us to recast the partition function as a double integral over single-trace parameters with a specific Gaussian weight

$$\mathcal{Z} = \frac{N_1 N_2}{4\pi\nu\sqrt{r_1 r_2}} \int_{-\infty}^{\infty} dy_1 dy_2\ e^{\hat{\mathcal{F}}\left[\lambda_1 - Cy_1 + Sy_2, N_1^2\right] + \hat{\mathcal{F}}\left[\lambda_2 - \frac{1}{\nu}(Sy_1 + Cy_2), N_2^2\right]}\, e^{\frac{N^2}{4}\left[\frac{y_1^2}{r_1} + \frac{y_2^2}{r_2}\right]}\ . \tag{2.11}$$

It is now possible to rewrite $\mathcal{Z}$ in a more explicit form thanks to the well-known solution of the one-matrix model [39, 38]. For the $k = 2$ multi-critical matrix model, in particular, the free energy reads

$$\hat{\mathcal{F}}[\lambda, N^2] = N^2\left(-a_1 x + \frac{1}{2}a_2 x^2\right) + F(x, N^2)\,, \tag{2.12}$$

where

$$\begin{aligned} F(x, N^2) = \ & N^2\left(-\frac{2}{5}a_3 x^{5/2} + \ldots\right) + N^0\left(-\frac{1}{24}\log x + \ldots\right) + \\ & + N^{-2}\left(a_4 x^{-5/2} + \ldots\right) + \mathcal{O}(N^{-4})\,, \end{aligned} \tag{2.13}$$

with

$$x = c_2 + \lambda\,, \quad a_1 = 4\,, \quad a_2 = 576\,, \quad a_3 = 6144\sqrt{3}\,, \quad c_2 = \frac{1}{48}\,, \quad \cdots \tag{2.14}$$

The omitted terms inside each parenthesis in the expression of $F$ (2.13) will be subleading in the double-scaling limit at each order in $N$ and will not play a rôle in our discussion.

Our case involves the free energy of two decoupled one-matrix models with parameters $x_1$, $x_2$. These parameters, which are to be integrated over in (2.11), are related to the variables $y_1, y_2$ by the linear transformation

$$y_1 = \Delta_1 - Cx_1 - Sx_2\,, \quad \Delta_1 = C\Lambda_1 + \nu S\Lambda_2 \tag{2.15a}$$

$$y_2 = \Delta_2 + Sx_1 - Cx_2\,, \quad \Delta_2 = -S\Lambda_1 + \nu C\Lambda_2\,, \tag{2.15b}$$

where we define

$$\Lambda_1 = c_2 + \lambda_1\,, \quad \Lambda_2 = c_2 + \lambda_2\ . \tag{2.16}$$

Inserting this information into (2.11) and dropping an unimportant overall constant factor we obtain the partition function

$$\mathcal{Z} = N^2 \int_{-\infty}^{\infty} dx_1 dx_2\; e^{\frac{N^2}{2}\left[\left(-2a_1 - \frac{\Delta_1 C}{r_1} + \frac{\Delta_2 S}{r_2}\right)x_1 + \left(-2a_1\nu - \frac{\Delta_1 S}{r_1} - \frac{\Delta_2 C}{r_2}\right)x_2\right]} \times \tag{2.17}$$
$$e^{\frac{N^2}{4}\left[\left(2a_2 + \frac{C^2}{r_1} + \frac{S^2}{r_2}\right)x_1^2 + \left(2a_2 + \frac{S^2}{r_1} + \frac{C^2}{r_2}\right)x_2^2 + 2CS\left(\frac{1}{r_1} - \frac{1}{r_2}\right)x_1 x_2\right]}\; e^{F(x_1, N_1^2) + F(x_2/\nu, N_2^2)}\ .$$

To diagonalize the quadratic term in the exponent of the integrand we rotate from $(x_1, x_2)$ to $(x_+, x_-)$:

$$x_1 = U_1^+ x_+ + U_1^- x_-\,, \quad x_2 = U_2^+ x_+ + U_2^- x_-\,, \tag{2.18}$$

where

$$U_1^+ = S\,, \quad U_1^- = C\,, \quad U_2^+ = C\,, \quad U_2^- = -S\ . \tag{2.19}$$

In terms of the rotated variables

$$\mathcal{Z} = N^2 \int_{-\infty}^{\infty} dx_+ dx_- e^{\frac{N^2}{2}(E^+ x_+ + E^- x_-)} e^{-N^2 m_+^2 x_+^2 - N^2 m_-^2 x_-^2} \times$$
$$e^{F(U_1^+ x_+ + U_1^- x_-, N_1^2) + F((U_2^+ x_+ + U_2^- x_-)/\nu, N_2^2)}\,, \tag{2.20}$$

where

$$E^{\pm} = \left(-2a_1 - \frac{\Delta_1 C}{r_1} + \frac{\Delta_2 S}{r_2}\right) U_1^{\pm} - \left(2a_1\nu + \frac{\Delta_1 S}{r_1} + \frac{\Delta_2 C}{r_2}\right) U_2^{\pm}\,, \tag{2.21a}$$

$$m_{\pm}^2 = -\frac{1}{8}\left(\frac{1}{r_1} + \frac{1}{r_2} + 4a_2 \pm \frac{|r_1 - r_2|}{r_1 r_2}\right)\ . \tag{2.21b}$$

Depending on the sign of the saddle point mass squared parameters $m_{\pm}^2$ we can define a variety of double scaling limits. We will distinguish between the following alternatives: $(a)$ both $m_{\pm}^2$ are positive, $(b)$ one mass$^2$ is positive, the other is zero, $(c)$ both masses are zero, or $(d)$ at least one mass is tachyonic.

($a$) occurs if and only if

$$\frac{1}{r_1} < -2a_2\,, \quad \frac{1}{r_2} < -2a_2\ . \tag{2.22}$$

In this case, the only sensible double scaling limit that we can take requires

$$N \to \infty\,, \quad E^{\pm} \to 0\,, \quad \text{so that } \frac{E^{\pm}}{4m_{\pm}^2} N^{4/5} a_3^{2/5} = t_{\pm} \text{ is kept fixed .} \tag{2.23}$$

After standard manipulations (see [24]), we deduce the partition function

$$\mathcal{Z} = e^{F(U_1^+ t_+ + U_1^- t_-) + F(\nu^{-1/5}(U_2^+ t_+ + U_2^- t_-))} = e^{F(t_1)+F(t_2)} = \mathcal{Z}(t_1)\mathcal{Z}(t_2) \tag{2.24}$$

with the obvious definition of the scaling parameters $t_1$ and $t_2$. The resulting theory is the product of two undeformed, decoupled matrix models, which describe the union of two decoupled $(2,3)$ minimal strings. The double scaling limit has driven the theory back to the undeformed point where the effects of the double-trace deformation are washed away. A similar effect in the case of a single matrix model was observed in [24].

This behavior is reminiscent of what happens with an irrelevant perturbation in a higher dimensional quantum field theory when we follow the renormalization group flow towards the infrared. In our matrix model example, going towards the infrared is achieved by the double scaling limit which zooms around a critical point. When the condition (2.22) holds, the effects of the double-trace deformation (2.5) disappear in the double scaling limit and the perturbation behaves as an irrelevant operator. In fact, this is more than an analogy: in higher dimensions there is a range of parameters where double-trace perturbations generalizing (2.5) are indeed irrelevant in the usual RG sense [22].

Case ($b$) is more interesting. Now $m_+^2 = 0$ and $m_-^2 > 0$, which is equivalent to

$$\left(\frac{1}{r_1} + 2a_2\right)\left(\frac{1}{r_2} + 2a_2\right) = 0\ . \tag{2.25}$$

This condition is satisfied by a two-parameter family of deformations. Notice that under (2.25) it is impossible to achieve a non-vanishing $g_{12}$ without turning on at the same time the other two couplings $g_{11}, g_{22}$.

To see what happens to the free energy, we set $m_+^2 = 0$ in (2.20), and take the double scaling limit

$$\begin{aligned} &N \to \infty\,, \qquad E^{\pm} \to 0\,, \qquad t_+ = x_+ N^{4/5} a_3^{2/5} \text{ fixed}\,, \\ &t_- = \frac{E^-}{4m_-^2} N^{4/5} a_3^{2/5} \text{ fixed}\,, \qquad \tilde{t}_+ = \frac{E^+}{2} N^{6/5} a_3^{-2/5} \text{ fixed}\,. \end{aligned} \tag{2.26}$$

After standard manipulations the double scaled partition sum becomes

$$\mathcal{Z}(\tilde{t}_+, t_-) = \int_{-\infty}^{\infty} dt_+\ e^{\tilde{t}_+ t_+ + F(U_1^+ t_+ + U_1^- t_-) + F(\nu^{-1/5}(U_2^+ t_+ + U_2^- t_-))}\ . \tag{2.27}$$

$\mathcal{Z}$ depends now on the double scaling parameters $\tilde{t}_+, t_-$ and parametrically on $\theta$, which is one of the deformation parameters (2.5).

Eq. (2.27) is an interesting exact formula that deserves a few comments. First, we observe that the old scaling parameter $t_+$ has been transmuted to a new scaling parameter $\tilde{t}_+$. $t_+$ requires scaling with $N^{4/5}$, whereas $\tilde{t}_+$ requires scaling with $N^{6/5}$. A similar change of critical behavior occurs in a single hermitian matrix model deformed by a double-trace operator as was pointed out originally in [23, 24]. In the single matrix model case with a double-trace deformation of the form $g(\mathrm{Tr}\,\Phi^4)^2$, the new critical behavior occurs when $g = -\frac{1}{2a_2}$ and exhibits the partition function

$$\mathcal{Z}(\tilde{t}) = \int_{-\infty}^{\infty} dt\ e^{\tilde{t}t+F(t)}\ . \tag{2.28}$$

Eq. (2.27) generalizes this effect to the two-matrix example (2.3).

Both (2.27) and (2.28) are bilateral Laplace transforms of the partition sum of the original undeformed theory. For non-vanishing $g_{12}$ (2.27) is *not* anymore the partition function of the decoupled product of two one-matrix models. The dual minimal string interpretation of this formula will be discussed in section 4. For $g_{12} = 0$ we expect to recover the partition function of a decoupled product of theories. This partition function is factorizable with one factor being the double scaled partition function of an undeformed one-matrix model and the other being the Laplace transformed partition function as in (2.28). Indeed, we can check that when $g_{12} = 0$

$$\mathcal{Z} = e^{F(\nu^{-1/5}t_-)} \int_{-\infty}^{\infty} dt_+ e^{\tilde{t}_+ t_+ + F(t_+)}\ . \tag{2.29}$$

The non-perturbative meaning of eqs. (2.27)–(2.29) is not completely clear. For instance, there are well-known problems in defining the one-matrix free energies $F(t)$ non-perturbatively. Presumably, we can avoid this problem by looking at a different class of matrix models, which are non-perturbatively well-defined (see *e.g.* [40] and references therein). Even then, however, we have to check the convergence of the integral appearing in the Laplace transform or whatever generalizes the Laplace transform. Despite these important potential issues, eqs. (2.27)–(2.29) work well at any order in perturbation theory, which is what we will focus on in this paper.

As a final comment on this case, we point out that there is an interesting analog of the critical behavior captured by eqs. (2.27), (2.29) in higher dimensional AdS/CFT examples [22]. In conformal perturbation theory one finds again a one-parameter family of interacting fixed points with non-vanishing double-trace couplings, which translate in the AdS/CFT correspondence to a system of coupled string theories on a union of AdS spaces.

When both masses in (2.21b) are taken to be zero (this is case ($c$) above) we obtain

$$g_{12} = 0 \text{ and } g_{11} = g_{22} = -\frac{1}{2a_2} \text{ or } 0\ . \tag{2.30}$$

With $g_{11} = g_{22} = -\frac{1}{2a_2}$ we are dealing with the decoupled product of two one-matrix models tuned to their individual double-trace deformed critical points that were analyzed in [24] (see eq. (2.28)).

Finally, when at least one of the masses $m_\pm$ is tachyonic, the theory goes into a branched polymer phase where we cannot define anymore a double scaling limit describing a sum over continuous surfaces [24].

**Case $(ii)$: $r_1 r_2 = 0$.** Similar manipulations can be performed when the determinant of the matrix of double-trace parameters $g_{ij}$ vanishes, *i.e.* when $r_1 r_2 = 0$. In this case, there is only one integration over single-trace deformation parameters in (2.11) and one can obtain again different critical behaviors depending on the sign of the saddle point mass squared $m^2$. In the stability region $m^2 > 0$, we recover (2.24), the decoupled product of two one-matrix models. When the mass $m$ is zero, which occurs for a one-parameter family of double-trace deformations parametrized by the angle $\theta$, we recover the interacting product of one-matrix models (2.27). When the mass is tachyonic the theory goes into a branched polymer phase.

### 2.1.2 Comments on the general $\mathcal{M}_{2,2k-1} \otimes \mathcal{M}_{2,2p-1}$ product

Much of what we said above about the $\mathcal{M}_{2,3} \otimes \mathcal{M}_{2,3}$ product goes through unchanged to the more general $\mathcal{M}_{2,2k-1} \otimes \mathcal{M}_{2,2p-1}$ product of one-matrix models, although some important changes in the conclusion occur when $k \neq p$. In the general case, the double-trace deformed partition function reads

$$\mathcal{Z} = \int D\Phi_1 D\Phi_2 e^{-N\left[\operatorname{Tr} V_k(\Phi_1) + (c_2+\lambda_1)\operatorname{Tr}\Phi_1^4 + \operatorname{Tr} V_p(\Phi_2) + (c_2+\lambda_2)\operatorname{Tr}\Phi_2^4\right]} \times \\ e^{-\left[g_{11}(\operatorname{Tr}\Phi_1^4)^2 + g_{22}(\operatorname{Tr}\Phi_2^4)^2 + 2g_{12}\operatorname{Tr}\Phi_1^4 \operatorname{Tr}\Phi_2^4\right]} \qquad (2.31)$$

with the potential $V_k$ still defined as in (2.2).

We recall that the free energy of the $k$-th multicritical matrix model is

$$\hat{\mathcal{F}}[\lambda] = N^2\left(-a_1 x + \frac{1}{2}a_2 x^2\right) + F_k(x, N^2)\,, \qquad (2.32)$$

where as before $x = c_2 + \lambda$, but now more generally

$$F_k(x, N^2) = -\frac{k}{2k+1} a_3 N^2 x^{(2k+1)/k} + \ldots \qquad (2.33)$$

Because of the $k$ dependence of the non-singular part of the free energy $F_k(x, N^2)$, the double scaling limit in the single-trace theory is

$$N \to \infty\,, \quad x \to 0\,, \quad t = x N^{2k/(2k+1)} \text{ fixed}\,. \qquad (2.34)$$

On the other hand, since the singular part of the free energy in (2.32) is $k$-independent, all the manipulations regarding this part go through unchanged as in the $\mathcal{M}_{2,3} \otimes \mathcal{M}_{2,3}$ case. In particular, the criteria for stability or instability remain the same.

Important differences occur when we take double scaling limits. There are no substantial changes when $k = p$, in which case the expressions we derived in the $\mathcal{M}_{2,3} \otimes \mathcal{M}_{2,3}$ case remain true with the obvious modifications of double scaling limits. On the other hand, when $k \neq p$ we find that that there is no sensible double scaling limit that leads to the analog of eq. (2.27). The reason for the absence of a sensible double scaling limit is the difference between the scaling behaviors (2.34) for $k \neq p$. We observe a similar effect in conformal perturbation theory in higher dimensional QFT examples [22]. New fixed points with non-zero double-trace coupling $g_{12}$ exist only when the single-trace operators $\mathcal{O}_1, \mathcal{O}_2$ participating in the double-trace deformation have the same scaling dimension.

### 2.1.3 More general deformations of the $\mathcal{M}_{2,2k-1} \otimes \mathcal{M}_{2,2k-1}$ product

For a single $k$-th matrix model new double scaling limits can be defined by adding double-trace deformations $g\mathcal{O}^2$ with a general scaling operator $\mathcal{O}$ after the necessary fine-tuning of $g$ [24]. In that case, one obtains the free energy

$$\tilde{\mathcal{F}}(t, \tilde{t}_O) = \log \int_{-\infty}^{\infty} dt_{\mathcal{O}} \; e^{t_O \tilde{t}_O + \mathcal{F}(t, t_O)} \; . \tag{2.35}$$

This example can be generalized to

$$\tilde{\mathcal{F}}(\{\tilde{t}\}, \{T\}) = \log \prod_{i=1}^{n} \int_{-\infty}^{\infty} dt_i \; e^{\sum_{j=1}^{n} t_j \tilde{t}_j + \mathcal{F}(\{t\},\{T\})} \, , \tag{2.36}$$

where $\{T\}$ is the set of coupling constants that remain unintegrated.

Accordingly, for a product of $k$-th matrix models new double scaling limits can be defined by tuning a set of multi-trace parameters. Consider, for instance, two $k$-th one-matrix models deformed by the double trace operator $g_{11}\mathcal{O}_1^2 + g_{22}\mathcal{O}_2^2 + 2g_{12}\mathcal{O}_1\mathcal{O}_2$, where $\mathcal{O}_1, \mathcal{O}_2$ are respectively scaling operators in the matrix models 1 and 2. We expect new double scaling limits leading to the free energies

$$\tilde{\mathcal{F}}(t_1, t_2, \tilde{\tau}_+, \tau_-) = \log \int_{-\infty}^{\infty} d\tau_+ \; e^{\tau_+ \tilde{\tau}_+ + \mathcal{F}_k(t_1, U_1^+ \tau_+ + U_1^- \tau_-) + \mathcal{F}_k(t_2, U_2^+ \tau_+ + U_2^- \tau_-)} \, , \tag{2.37}$$

where $t_1, t_2$ are single-trace couplings for the lowest dimension operators. Further generalizations can be envisioned by introducing more couplings.

## 2.2 Matrix quantum mechanics and $c = 1$ string theories

Analogous statements can be made for a pair of matrix quantum mechanics theories coupled by a double-trace deformation. In appendix A we consider in some detail double-trace deformations involving the cubic single-trace operator $\operatorname{Tr}\Phi^3$. The partition function of the theory we analyze there is

$$\begin{aligned} \mathcal{Z} = \int D\Phi_1(t) D\Phi_2(t) \; & e^{-N \int_0^{2\pi R} dt [\operatorname{Tr}(\frac{1}{2}\dot{\Phi}_1^2 + \frac{1}{2}\Phi_1^2 - \lambda_1 \Phi_1^3) + \operatorname{Tr}(\frac{1}{2}\dot{\Phi}_2^2 + \frac{1}{2}\Phi_2^2 - \lambda_2 \Phi_2^3)]} \times \\ & e^{-\int_0^{2\pi R} dt [g_{11} (\operatorname{Tr}\Phi_1^3)^2 + 2 g_{12} \operatorname{Tr}\Phi_1^3 \operatorname{Tr}\Phi_2^3 + g_{22} (\operatorname{Tr}\Phi_2^3)^2]} \; . \end{aligned} \tag{2.38}$$

The matrix model lives on a compact one-dimensional spacetime with radius $R$. Besides this extra feature (that can be dealt with a Fourier transformation) most of the elements of the analysis of the previous subsections go through also in this case. The possible double scaling limits and the resulting expressions are analogous to the zero-dimensional matrix model case above, so we will omit a detailed discussion here. The interested reader can find a detailed analysis of the partition function (2.38) in appendix A.

## 2.3 Triple intersections and beyond

So far we have restricted our attention to pairs of matrix models. In a suitable double scaling limit, these models provide the holographic description of a product string theory with two components. There is a subtle communication between the individual components of this product induced by the double-trace deformation of the matrix model dual. There are various possible generalizations of this picture.

We could consider, for instance, $n$ one-matrix models coupled together by a general multi-trace interaction. We can imagine a large variety of possibilities. One of them is to take $n$ theories coupled two-by-two via double-trace deformations. The overall theory is an $n$-matrix model with an action of the form

$$S = \sum_{i=1}^{n} S_i + \sum_{i,j=1}^{n} g_{ij} \mathcal{O}_i \mathcal{O}_j \, , \tag{2.39}$$

where $S_i$ are the single-trace actions and $\mathcal{O}_i$ are single-trace operators, *e.g.* $\mathcal{O}_i = \mathrm{Tr}\, \Phi_i^4$. Recasting $g_{ij}$ in terms of $n$ eigenvalues and an $O(n)$ matrix parametrized by $n$ orthogonal unit vectors $\vec{v}^a$, we can write $S$ as

$$S = \sum_{i=1}^{n} S_i + \sum_{a=1}^{n} r_a \left( \sum_{i=1}^{n} \vec{v}_i^a \mathcal{O}_i \right)^2 \, . \tag{2.40}$$

The corresponding partition function can be analyzed as above and allows for an obvious extension of the double scaling limits presented in previous subsections provided that the operators $\mathcal{O}_i$ have compatible scaling behaviors.

Other possibilities include higher multi-trace interactions with three or more single-trace components. This allows for a more intricate set of double scaling limits. As an illustration, in appendix B we analyze the triple intersection of three 2nd multicritical matrix models with the action

$$S = \sum_{i=1}^{n} S_i + \frac{1}{N} \sum_{i,j,k=1}^{3} g_{ijk} (\mathrm{Tr}\, \Phi_i^4)(\mathrm{Tr}\, \Phi_j^4)(\mathrm{Tr}\, \Phi_k^4) \, . \tag{2.41}$$

For example, in the special case, where $g_{123}$ is the only non-vanishing triple-trace coupling, we can find a double scaling limit with partition function

$$\mathcal{Z}(\tilde{t}^1, \tilde{t}^2, t_3) = \int_{-\infty}^{\infty} dt_1 dt_2 \; e^{\tilde{t}^1 t_1 + \tilde{t}^2 t_2 + \sum_{i=1}^{3} F(\mathcal{U}_i^1 t_1 + \mathcal{U}_i^2 t_2 + \mathcal{U}_i^3 t_3)} \, . \tag{2.42}$$

$\mathcal{U}_i^j$ are constants that can be determined. Once again we obtain a modified partition function related to the undeformed one by a Laplace transform. A more intricate pattern of double scaling limits is expected for quartic and higher multi-trace deformations.

One qualitatively new feature in the triple-trace example, compared to the double-trace case, is the possibility to find a non-trivial fixed point with $g_{123} \neq 0$ and no other multi-trace couplings turned on. This was not possible in the case of double-trace deformations, where at points with $g_{11} = g_{22} = 0, g_{12} \neq 0$ at least one of the saddle point masses $m_\pm$ was tachyonic.

In higher dimensional QFTs unitarity and the requirement that the $n$-trace interaction is a relevant or marginal operator puts an upper bound on $n$. For example, in $d = 4$ dimensions the bound is $n = 3$. The bound is absent only in two or lower dimensions. The general structure of the one-loop $\beta$-functions for multi-trace interactions in $d$ dimensions will be discussed in [22].

## 3. Correlation functions in coupled minimal string theories

In this section we will discuss scattering amplitudes in a theory of coupled minimal strings using the matrix model definition (2.27). We will illustrate the main points by recasting the correlation functions of the deformed product theory as a diagrammatic expansion in terms of correlation functions in the undeformed theory. As an example, we will consider two-point functions at tree-level and one-loop. The analysis in this section is a simple extension of the corresponding study of correlation functions in a single modified minimal string in [41]. It will provide a concrete illustration of the non-local string theory structure underlying the dynamics of the string theories of interest and will be a good guide for the higher dimensional AdS/CFT examples discussed in the introduction, where it is much harder to explore this structure explicitly beyond tree-level.

To be definite, let us focus on one of the cases discussed in section 2: two $(2, 2k-1)$ minimal strings coupled on the matrix model side by a double-trace deformation. The partition function of the modified theory is given by the Laplace transform

$$e^{\tilde{F}(\tilde{t}_+, t_-; t_1, t_2)} = \int_{-\infty}^{\infty} dt_+ \; e^{\lambda^{-1}\tilde{t}_+ t_+ + F_1(St_+ + Ct_-, t_1) + F_2(Ct_+ - St_-, t_2)} \; . \tag{3.1}$$

$t_1, t_2$ are single-trace couplings for the scaling operators $\mathcal{O}_1$, $\mathcal{O}_2$ of theory 1 and 2 respectively, whose correlations functions we will compute. These operators do not partake in the double-trace deformation that couples the two minimal strings. The constant $\lambda$ has been inserted in (3.1) to help us keep track of the genus expansion of the free energies.

We can obtain the all-genus diagrammatic expansion of correlation functions in the deformed theory by expanding the integral expression (3.1) around the saddle point value of $t_+$, which we will call $t_s$. $t_s$ is determined by the following equation

$$\tilde{t}_+ + S\langle \mathcal{P}_1 \rangle|_{St_s + Ct_-} + C\langle \mathcal{P}_2 \rangle|_{Ct_s - St_-} = 0 \, , \tag{3.2}$$

where the subindex next to the one-point functions denotes the point at which the correlation functions are evaluated. In what follows, we will keep this index implicit. $\mathcal{P}_1$ and $\mathcal{P}_2$ are operators in theory 1 and 2 respectively, whose single-trace couplings $St_+ + Ct_-$, $Ct_+ - St_-$ appear inside the integral in (3.1). In (3.2) we used the $n = 1$ version of the identities

$$\lambda^{-1} \langle \mathcal{P}_i^n \rangle|_t = \partial_t^n F_i(t) \, , \quad i = 1, 2 \; . \tag{3.3}$$

The one-point functions in (3.2) receive contributions at all geni. Since we are interested in a genus expansion (in other words, an expansion in powers of $\lambda$) of $\tilde{F}$, it will be useful to define our point of expansion by the tree-level version of (3.2)

$$\tilde{t}_+ + S\langle \mathcal{P}_1 \rangle_0 + C\langle \mathcal{P}_2 \rangle_0 = 0 \; . \tag{3.4}$$

The subindex in correlation functions will denote from now on the genus.

Setting $t_+ = t_s + t$ for the integration variable in (3.1), we expand around $t_s$ to obtain the free energy expression

$$\begin{aligned}\tilde{F}(\tilde{t}_+, t_-; t_1, t_2) \;=\; & \lambda^{-1}\tilde{t}_+ t_s + F_1(St_s + Ct_-, t_1) + F_2(Ct_s - St_-, t_2) + \\ & + \log \int_{-\infty}^{\infty} dt \;\; \exp\left[ t \sum_{g=1}^{\infty} \lambda^{g-1} \left( S\langle \mathcal{P}_1 \rangle_g + C \langle \mathcal{P}_2 \rangle_g \right) \right. \\ & \left. + \sum_{n=2}^{\infty} \sum_{g=0}^{\infty} \frac{1}{n!} t^n \lambda^{g-1} \left( S^n \langle \mathcal{P}_1^n \rangle_g + C^n \langle \mathcal{P}_2^n \rangle_g \right) \right] .\end{aligned} \tag{3.5}$$

For example, from this expression we read off the tree-level and one-loop free energies

$$\tilde{F}_{\text{tree}}(\tilde{t}_+, t_-; t_1, t_2) \;=\; \lambda^{-1}\tilde{t}_+ t_s + F_1^{(0)}(St_s + Ct_-, t_1) + F_2^{(0)}(Ct_s - St_-, t_2) \,, \tag{3.6a}$$

$$\begin{aligned}\tilde{F}_{one-loop}(\tilde{t}_+, t_-; t_1, t_2) \;=\; & F_1^{(1)}(St_s + Ct_-, t_1) + F_2^{(1)}(Ct_s - St_-, t_2) \\ & -\frac{1}{2} \log \left[ S^2 \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 + C^2 \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0 \right] \,,\end{aligned} \tag{3.6b}$$

where $F_i = \sum_{g=0}^{\infty} \lambda^{g-1} F_i^{(g)}$ $(i = 1, 2)$ is the genus expansion of the free energy of each theory. We see that the tree-level expression (3.6a) is merely a Legendre transform of the total undeformed free energy. On the other hand, the one-loop expression (3.6b) receives contributions both from tree-level and one-loop quantities in the undeformed theory.

At any genus, correlation functions can be computed with the use of the Feynman diagrams in figure 1, which include vertices (tadpoles, 2- and higher $n$-point vertices) of both minimal strings. Vertices of either string can be connected with a common propagator. The strength of the vertices and the actual form of the propagator is controlled by the double-trace parameter $\theta$. The generic diagram of the modified theory is a sum of correlation functions in the original product theory on disconnected worldsheets. One could reformulate the new interactions in terms of a set of non-local interactions on the worldsheet of the original theory. Because of its tractability, our setup provides a concrete example of the non-local string theory construction proposed in [19]. We will return to this point in the next section.

To illustrate the generic structure of correlation functions in the modified product theory with a few examples, we will now consider the two-point functions of the operators $\mathcal{P}_i$, $\mathcal{O}_i$ $(i = 1, 2)$ at tree-level and one-loop.

### 3.1 Two-point functions at tree-level

At tree-level the free energy of the modified theory is the Legendre transform of the original theory. We can obtain the tree-level amplitudes of the scaling operators $\mathcal{O}_1$, $\mathcal{O}_2$ by differentiating (3.6a) with respect to $t_1$, $t_2$ and using the chain rule. For two-point functions we find (we will denote the modified 2-point functions as $\langle\langle \cdots \rangle\rangle$)

$$\langle\langle \mathcal{O}_1 \mathcal{O}_1 \rangle\rangle_0 = \langle \mathcal{O}_1 \mathcal{O}_1 \rangle_0 + \frac{S^2 \langle \mathcal{O}_1 \mathcal{P}_1 \rangle_0^2}{S^2 \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 + C^2 \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0} \,, \tag{3.7a}$$

$$\langle\langle \mathcal{O}_2 \mathcal{O}_2 \rangle\rangle_0 = \langle \mathcal{O}_2 \mathcal{O}_2 \rangle_0 + \frac{C^2 \langle \mathcal{O}_2 \mathcal{P}_2 \rangle_0^2}{S^2 \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 + C^2 \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0} \,, \tag{3.7b}$$

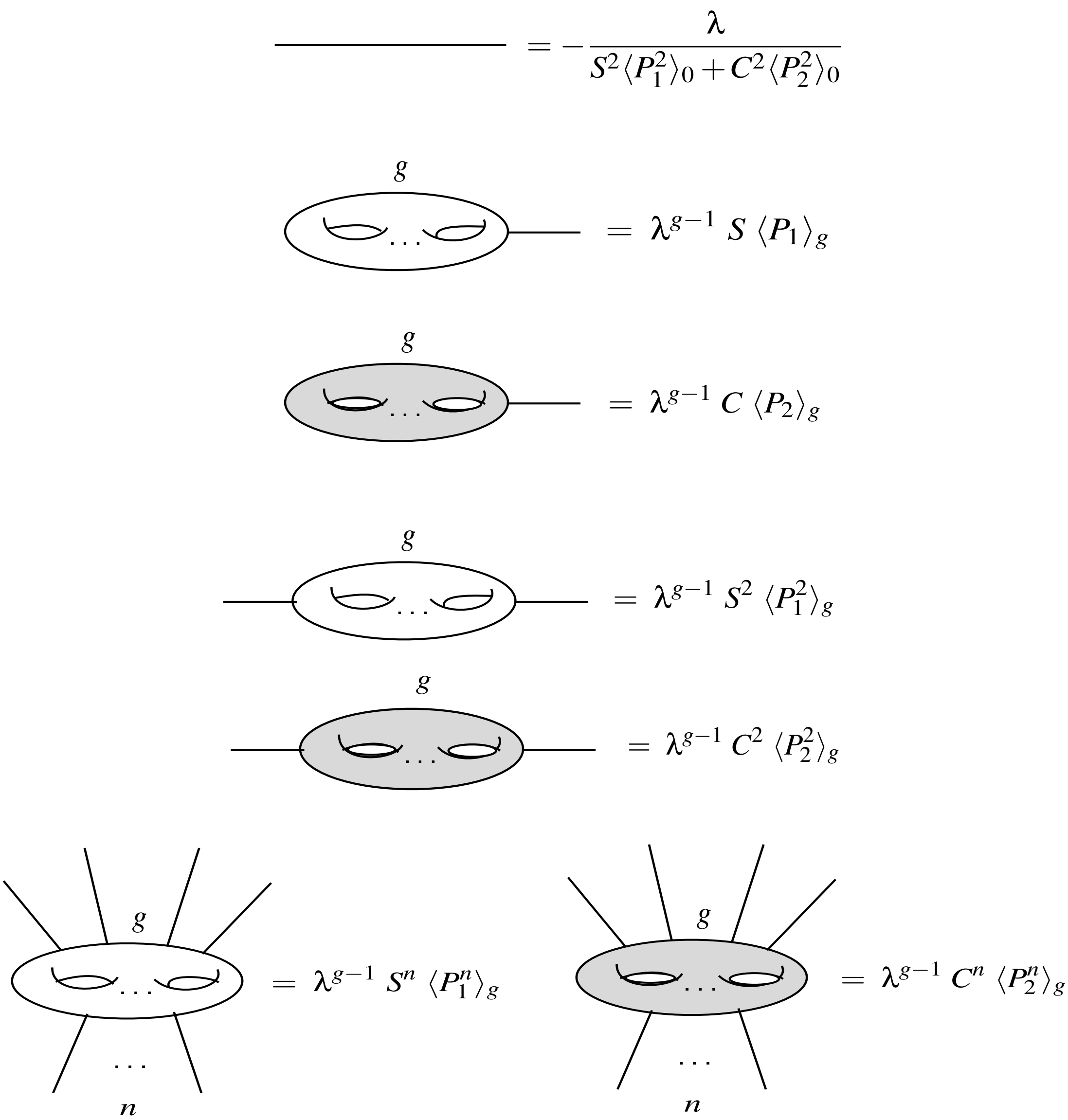
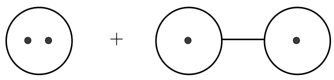
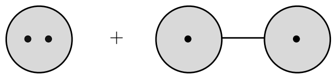


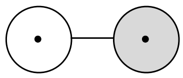

**Figure 1:** Vertices necessary for the diagrammatic expansion of the modified free energy of two coupled minimal strings. The white (grey) worldsheets are respectively worldsheets of minimal string 1 (2). The vertices in the last line apply to any genus $g = 0, 1, 2, \ldots$ and can have $n = 3, 4, 5, \ldots$ external legs.

$$\langle\langle \mathcal{O}_1 \mathcal{O}_2 \rangle\rangle_0 = \frac{SC \langle \mathcal{O}_1 \mathcal{P}_1 \rangle_0 \langle \mathcal{O}_2 \mathcal{P}_2 \rangle_0}{S^2 \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 + C^2 \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0} \ . \tag{3.7c}$$

As expected, the last correlation function that describes the non-trivial interaction between the two minimal strings is proportional to $SC$, which vanishes only when the double-trace coupling $g_{12}$ is zero. Diagrammatically, these two-point functions can be recast as

$$\langle\langle \mathcal{O}_1 \mathcal{O}_1 \rangle\rangle_0 =$$  , (3.8a)

$$\langle\langle \mathcal{O}_2 \mathcal{O}_2 \rangle\rangle_0 =$$  , (3.8b)

$$\langle\langle \mathcal{O}_1 \mathcal{O}_2 \rangle\rangle_0 =$$  . (3.8c)

Differentiating the free energy $\tilde{F}$ with respect to the scaling parameters $\tilde{t}_+$, $t_-$ gives correlation functions for the operators $\tilde{\mathcal{P}}_+, \mathcal{P}_-$, which are defined in the following way. In the original theory, before the double-trace deformation, $\mathcal{P}_\pm$ are operators in the product theory defined by the linear combinations

$$\mathcal{P}_+ = S\mathcal{P}_1 + C\mathcal{P}_2 \, , \quad \mathcal{P}_- = C\mathcal{P}_1 - S\mathcal{P}_2 \, . \tag{3.9}$$

The couplings of $\mathcal{P}_\pm$ are what we called above $t_\pm$. In the deformed theory (3.1), $t_+$ is modified to $\tilde{t}_+$ and $t_-$ remains invariant. Accordingly, $\mathcal{P}_+$ is modified to $\tilde{\mathcal{P}}_+$ and $\mathcal{P}_-$ remains invariant.

At tree-level, $\tilde{\mathcal{P}}_+$ and $\mathcal{P}_-$ exhibit the one-point functions

$$\langle\langle \tilde{\mathcal{P}}_+ \rangle\rangle_0 = \lambda^{-1} t_s \, , \quad \langle\langle \mathcal{P}_- \rangle\rangle_0 = \lambda^{-1} \langle \mathcal{P}_- \rangle_0 \tag{3.10}$$

and the two-point functions

$$\langle\langle \tilde{\mathcal{P}}_+ \tilde{\mathcal{P}}_+ \rangle\rangle_0 = -\frac{1}{S^2 \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 + C^2 \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0} \, , \tag{3.11a}$$

$$\langle\langle \mathcal{P}_- \mathcal{P}_- \rangle\rangle_0 = C^2 \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 + S^2 \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0 - \frac{C^2 S^2 \left( \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 - \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0 \right)^2}{S^2 \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 + C^2 \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0} \, , \tag{3.11b}$$

$$\langle\langle \tilde{\mathcal{P}}_+ \mathcal{P}_- \rangle\rangle_0 = -\frac{CS \left( \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 - \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0 \right)}{S^2 \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 + C^2 \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0} \, . \tag{3.11c}$$

### 3.2 Two-point functions at one-loop

One-loop correlation functions can be obtained from (3.6b) in a similar fashion. Here we will list the two-point functions of the operators $\mathcal{O}_i$. Analogous expressions can be deduced for the amplitudes of the operators $\tilde{\mathcal{P}}_+$, $\mathcal{P}_-$. It will be convenient to set

$$\mathcal{K} = S^2 \langle \mathcal{P}_1 \mathcal{P}_1 \rangle_0 + C^2 \langle \mathcal{P}_2 \mathcal{P}_2 \rangle_0 \, . \tag{3.12}$$

The two-point function $\langle\langle\mathcal{O}_1\mathcal{O}_1\rangle\rangle_1$ reads

$$\begin{aligned}
\langle\langle\mathcal{O}_1\mathcal{O}_1\rangle\rangle_1 = {} & \langle\mathcal{O}_1\mathcal{O}_1\rangle_1 + S^2\mathcal{K}^{-1}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_1\mathcal{O}_1\rangle_1 - S^6\mathcal{K}^{-3}\langle\mathcal{P}_1\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_1^3\rangle_0 - \\
& -C^3S^3\mathcal{K}^{-3}\langle\mathcal{P}_1\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_2^3\rangle_0 - CS^5\mathcal{K}^{-3}\langle\mathcal{P}_2\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_1^3\rangle_0 \\
& -C^3S^2\mathcal{K}^{-3}\langle\mathcal{P}_2\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_2^3\rangle_0 + S^4\mathcal{K}^{-2}\langle\mathcal{P}_1^2\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2 + \\
& +CS^3\mathcal{K}^{-2}\langle\mathcal{P}_2^2\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2 + S^2\mathcal{K}^{-1}\langle\mathcal{P}_1\rangle_1\langle\mathcal{P}_1\mathcal{O}_1^2\rangle_0 + CS\mathcal{K}^{-1}\langle\mathcal{P}_2\rangle_1\langle\mathcal{P}_1\mathcal{O}_1^2\rangle_0 + \\
& +\frac{1}{2}S^4\mathcal{K}^{-2}\langle\mathcal{P}_1^2\mathcal{O}_1\rangle_0 + S^6\mathcal{K}^{-3}\langle\mathcal{P}_1^2\mathcal{O}_1\rangle_0\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_1^3\rangle_0 + \\
& +C^3S^3\mathcal{K}^{-3}\langle\mathcal{P}_1^2\mathcal{O}_1\rangle_0\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2^3\rangle_0 - \frac{1}{2}S^2\mathcal{K}^{-1}\langle\mathcal{P}_1^2\mathcal{O}_1^2\rangle_0 - \\
& -S^4\mathcal{K}^{-2}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_1^3\mathcal{O}_1\rangle_0 - \frac{1}{2}S^4\mathcal{K}^{-2}\langle\mathcal{P}_1^3\rangle_0\langle\mathcal{P}_1\mathcal{O}_1^2\rangle_0 - \\
& -\frac{1}{2}CS^3\mathcal{K}^{-2}\langle\mathcal{P}_2^3\rangle_0\langle\mathcal{P}_1\mathcal{O}_1^2\rangle_0 - \frac{1}{2}S^6\mathcal{K}^{-3}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_1^4\rangle_0 - \\
& -\frac{1}{2}C^4S^2\mathcal{K}^{-3}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_2^4\rangle_0 + S^8\mathcal{K}^{-4}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_1^3\rangle_0^2 + \\
& +2C^3S^5\mathcal{K}^{-4}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_1^3\rangle_0\langle\mathcal{P}_2^3\rangle_0 + C^6S^2\mathcal{K}^{-4}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_2^3\rangle_0^2
\end{aligned} \tag{3.13}$$

and has the diagrammatic expansion in figure 2. A similar expansion, with obvious modifications, applies to $\langle\langle\mathcal{O}_2\mathcal{O}_2\rangle\rangle_1$.

The expansion of $\langle\langle\mathcal{O}_1\mathcal{O}_2\rangle\rangle_1$ is

$$\begin{aligned}
\langle\langle\mathcal{O}_1\mathcal{O}_2\rangle\rangle_1 = {} & CS\mathcal{K}^{-1}\langle\mathcal{P}_1\mathcal{O}_1\rangle_1\langle\mathcal{P}_2\mathcal{O}_2\rangle_0 + CS\mathcal{K}^{-1}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_1 - \\
& -CS^5\mathcal{K}^{-3}\langle\mathcal{P}_1\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_1^3\rangle_0 - C^2S^4\mathcal{K}^{-3}\langle\mathcal{P}_2\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_1^3\rangle_0 - \\
& -C^4S^2\mathcal{K}^{-3}\langle\mathcal{P}_1\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_2^3\rangle_0 - C^5S\mathcal{K}^{-3}\langle\mathcal{P}_2\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_2^3\rangle_0 - \\
& -C^2S^2\mathcal{K}^{-2}\langle\mathcal{P}_1\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2^2\mathcal{O}_2\rangle_0 - C^3S\mathcal{K}^{-2}\langle\mathcal{P}_2\rangle_1\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2^2\mathcal{O}_2\rangle_0 + \\
& +CS^3\mathcal{K}^{-2}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_1^2\rangle_1 + C^3S\mathcal{K}^{-2}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_2^2\rangle_1 + \\
& +CS^3\mathcal{K}^{-2}\langle\mathcal{P}_1\rangle_1\langle\mathcal{P}_1^2\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0 + C^2S^2\mathcal{K}^{-2}\langle\mathcal{P}_2\rangle_1\langle\mathcal{P}_1^2\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0 + \\
& +\frac{1}{2}C^2S^2\mathcal{K}^{-2}\langle\mathcal{P}_1^2\mathcal{O}_1\rangle_0\langle\mathcal{P}_2^2\mathcal{O}_2\rangle_0 - \frac{1}{2}CS^3\mathcal{K}^{-2}\langle\mathcal{P}_1^3\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0 \\
& -\frac{1}{2}C^3S\mathcal{K}^{-2}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2^3\mathcal{O}_2\rangle_0 + CS^7\mathcal{K}^{-4}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_1^3\rangle_0 \\
& +C^7S\mathcal{K}^{-4}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_2^3\rangle_0 + 2C^4S^4\mathcal{K}^{-4}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_1^3\rangle_0\langle\mathcal{P}_2^3\rangle_0 \\
& +C^2S^4\mathcal{K}^{-3}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2^2\mathcal{O}_2\rangle_0\langle\mathcal{P}_1^3\rangle_0 - \frac{1}{2}CS^5\mathcal{K}^{-3}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_1^4\rangle_0 - \\
& -\frac{1}{2}C^5S\mathcal{K}^{-3}\langle\mathcal{P}_1\mathcal{O}_1\rangle_0\langle\mathcal{P}_2\mathcal{O}_2\rangle_0\langle\mathcal{P}_2^4\rangle_0\ .
\end{aligned} \tag{3.14}$$

The corresponding diagrams appear in figure 3.

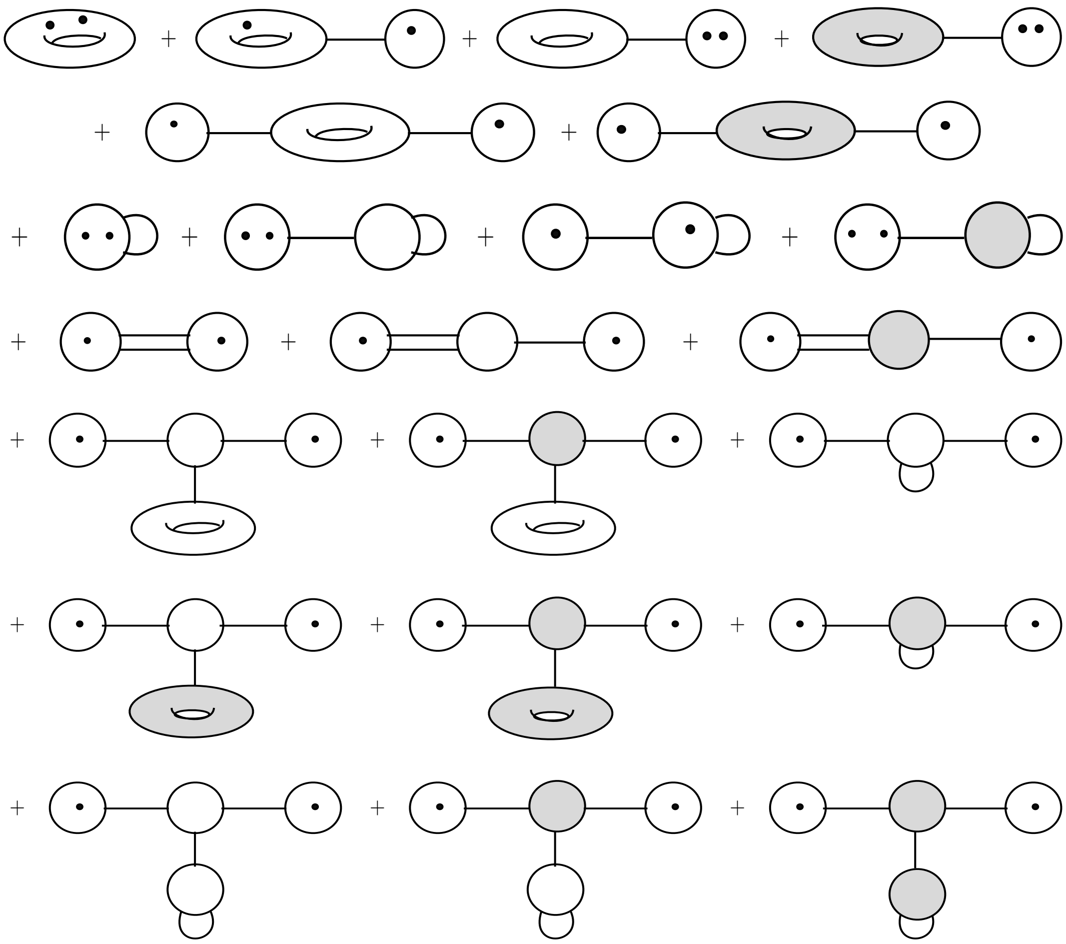


**Figure 2:** The diagrammatic expansion of the one-loop amplitude $\langle\langle\mathcal{O}_1\mathcal{O}_1\rangle\rangle_1$.

Most of the terms appearing in equations (3.13) and (3.14) have a unique representation as diagrams in figures 2 and 3. There are, however, some exceptions. These include the last three diagrams in figure 2 and their analogs in figure 3. For example, both diagrams in figure 4 can be associated to the amplitude contribution $\langle\mathcal{P}_1\mathcal{O}_1\rangle_0^2\langle\mathcal{P}_1^3\rangle_0^2$. Viewing the contact interactions as tiny wormholes that connect different parts of a worldsheet, we recognize these diagrams as representations of the same amplitude in a different region of the worldsheet moduli. In string perturbation theory one sums over all these configurations automatically.

There are many contributions to the one-loop renormalization of the two-point functions (3.8a)–(3.8c). Some of them are standard torus amplitudes in theory 1 or 2 separately, but the majority comes from contact interactions where worldsheets of theory 1 and/or 2 connect via a tiny 'neck' (wormhole) associated to a common propagator $\mathcal{K}^{-1}$ (3.12). Contact interactions between worldsheets of different theories are especially interesting since they provide a clear picture of how theories 1 and 2 communicate with

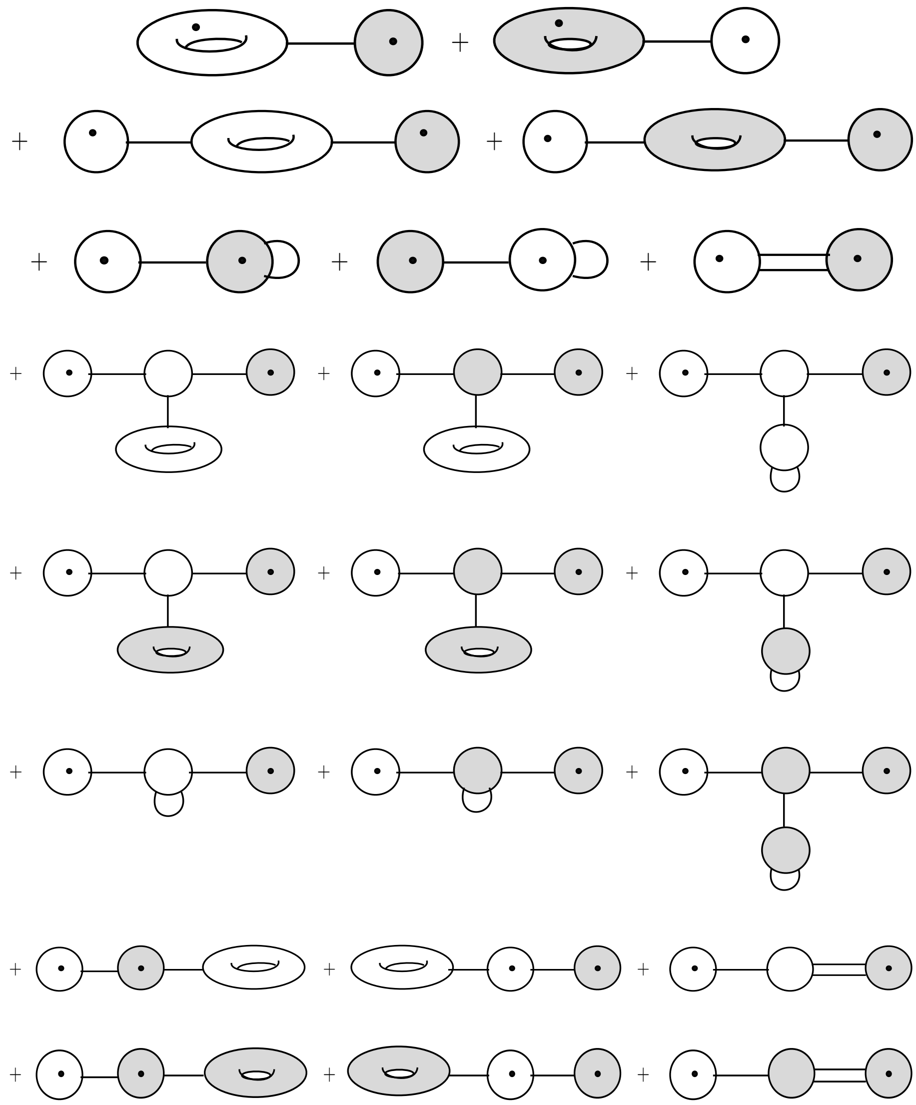


**Figure 3:** The diagrammatic expansion of the one-loop amplitude $\langle\langle\mathcal{O}_1\mathcal{O}_2\rangle\rangle_1$.

each other on the level of the worldsheet. Such interactions contribute both to single theory two-point functions ($\langle\langle\mathcal{O}_1\mathcal{O}_1\rangle\rangle$ or $\langle\langle\mathcal{O}_2\mathcal{O}_2\rangle\rangle$) and to mixed two-point functions of the form $\langle\langle\mathcal{O}_1\mathcal{O}_2\rangle\rangle$. In the latter case the 1-2 interactions work to renormalize the tree-level result (3.8c) either by renormalizing separately the single theory two-point functions $\langle\mathcal{O}_1\mathcal{P}_1\rangle_0$, $\langle\mathcal{O}_2\mathcal{P}_2\rangle_0$ or by renormalizing the propagator $\mathcal{K}^{-1}$ with a double wormhole interaction $C^2S^2\mathcal{K}^{-2}\langle\mathcal{P}_1^2\mathcal{O}_1\rangle_0\langle\mathcal{P}_2^2\mathcal{O}_2\rangle$ (the last diagram in the third line of figure 3). A similar diagram renormalizes the mixed two-point function $\langle g_1 g_2\rangle$ for the two AdS gravitons $g_1, g_2$ in the AdS/CFT example of [1, 2]. In that case, this is precisely the effect that lifts the mass of a certain linear combination of the gravitons and leads to massive gravity.

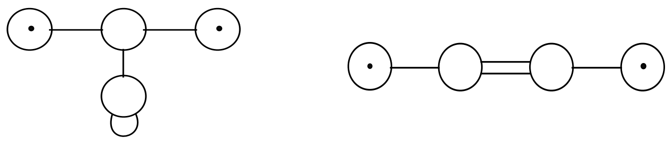


**Figure 4:** Two diagrams corresponding to the same amplitude contribution $\langle \mathcal{P}_1 \mathcal{O}_1 \rangle_0^2 \langle \mathcal{P}_1^3 \rangle_0^2$.

## 4. Worldsheet and spacetime aspects of the multi-trace coupling

In this section we want to discuss the worldsheet and spacetime interpretation of the theories defined above through the matrix model. There are two aspects we want to emphasize: (*a*) the modified genus expansion and non-locality on the worldsheet, and (*b*) the analogy with higher dimensional AdS/CFT examples and the possibility of an alternative formulation of the deformed theories that does not involve non-locality on the worldsheet.

### 4.1 Touching surfaces in Liouville gravity and the NLST structure

In conventional matrix models, *i.e.* matrix models with only single-trace interactions, the Feynman diagram expansion generates discretized random surfaces from which the continuous string worldsheet arises in a suitable double scaling limit. Multi-trace interactions modify this expansion by gluing together different random surfaces at a plaquette. In the continuum limit, one can think of this contact interaction as a tiny neck (wormhole) that creates a network of touching surfaces. Such microscopic degenerations of the string worldsheet are already present in the conventional theory without the multi-trace deformation, however, they give a sizable contribution that modifies the conventional result only for special values of the multi-trace couplings where a new double scaling limit is possible. The diagrammatic expansion of the previous section gives a clear demonstration of this effect of contact interactions between different worldsheets. In fact, the examples of section 3 give an even more dramatic illustration of this general phenomenon: even worldsheets of different string theories can touch via a common wormhole.

One can argue on general grounds that multi-trace interactions will have a similar effect also in the AdS/CFT correspondence in higher dimensions. String theories resulting from such a deformation were termed non-local string theories in [19], since one can presumably reproduce the effect of the tiny wormholes with a non-local deformation of the worldsheet action. More specifically, if $N_w$ is the number of disconnected components of the worldsheet $\Sigma = \Sigma^{(1)} \oplus \Sigma^{(2)} \oplus \cdots \oplus \Sigma^{(N_w)}$ one can envision a non-local worldsheet action of the form

$$\mathcal{S}_{ws} = \sum_{i=1}^{N_w} \int d^2\sigma^{(i)} \sqrt{g(\sigma)} \mathcal{L}_0 + \sum_{i,j=1}^{N_w} \int d^2\sigma_1^{(i)} d^2\sigma_2^{(j)} \sqrt{g((\sigma_1)g(\sigma_2)} \mathcal{G}[X(\sigma_1), X(\sigma_2)] + \\ + trilocal\ and\ higher - order\ interactions \tag{4.1}$$

where $X$ denotes collectively the target space fields and $\mathcal{G}$ is a non-local interaction. $\mathcal{G}$ and the rest of the higher order interactions in (4.1) should be determined directly in string theory by consistency, *e.g.* by requiring the cancellation of the Weyl anomaly. The precise rules for such theories have not been worked out, however, the consistency of the AdS/CFT correspondence implies that such non-local string theories should exist. In fact, we recognize a concrete example of such a theory in the non-critical NLSTs defined in section 2. In that case, see section 3, we can determine explicitly the precise rules for the diagrammatic expansion of the correlation functions and from them one could work backwards to deduce the required form of $\mathcal{G}$ *etc.*

### 4.2 An alternative interpretation?

When a CFT admits a dual supergravity description, it has been argued [25 – 29] that there is an alternative way to understand multi-trace deformations as mixed boundary conditions for the dual fields in AdS. In the matrix model case, there are indications for a similar reformulation of the deformation on the string theory side as a local effect on the worldsheet that involves changing the branch of the Liouville dressing of a vertex operator [23]. Here we will argue that there is a simple interpretation of this observation at tree-level, which is in accordance with our current understanding of Liouville theory and the analogous observations in the AdS/CFT correspondence [25 – 29]. Beyond tree-level, a similar reformulation does not go through, and one has to think of the deformed theory as an NLST along the lines of [19].

#### 4.2.1 The tree-level theory

To illustrate the main point, let us consider first the simpler situation of a single one-matrix model deformed by a double-trace deformation. The analogous AdS/CFT example involves a CFT deformed by a double-trace interaction of the form

$$\delta\mathcal{S} = \int d^d x \; g\mathcal{O}^2 \,, \tag{4.2}$$

where $\mathcal{O}$ is a single-trace operator with scaling dimension $\Delta$. $g$ is a coupling that scales like $N^0$. It will be useful to recall first the main features of this example.

If the scaling dimension $\Delta$ is less than $\frac{d}{2}$, the perturbing operator (4.2) is relevant and the theory runs towards an infrared (IR) fixed point, where $\mathcal{O}$ has a different scaling dimension $d-\Delta$. At large $N$, the IR theory is simply related to the UV theory by Legendre transform [42]. At tree-level and within the supergravity description, this RG running on the boundary has a holographic counterpart in $(d+1)$-dimensional AdS space as a flow between different boundary conditions for the bulk field $\phi$ which is dual to the operator $\mathcal{O}$.

There are two points we want to emphasize with respect to this statement. First, in contrast with the case of a single-trace perturbation, where the dual gravity background is deformed away from AdS, the multi-trace deformation does not backreact to the bulk geometry to leading order in $1/N$. Second, changing the boundary conditions of the bulk field does not mean that we pick a different solution of the bulk equations of motion for the dual field $\phi$. This is still controlled by regularity in the bulk. Instead, we modify the definition of the source — in other words, we modify the bulk/boundary dictionary. This last point will be important for the matrix model discussion below, so we take a moment to explain it here in slightly more detail (for a more detailed exposition of this well-known material we refer the reader to [43], which is what we will follow here mostly, and the standard references therein).

In coordinates where the Euclidean AdS metric takes the form

$$ds^2_{d+1} = \frac{1}{r^2}\left(dr^2 + \sum_{i=1}^{d}(dx^i)^2\right) \tag{4.3}$$

consider a scalar field $\phi$ with the action

$$S = \int d^d x dr \sqrt{g}\left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2\right) + \int d^d x \sqrt{g}\, \mathcal{L}_{\text{boundary}}|_{r=\epsilon} \ . \tag{4.4}$$

$\phi$ is the scalar field dual to the single-trace operator $\mathcal{O}$. The mass $m$ is related to the scaling dimension $\Delta$ via the relation

$$\Delta = \frac{d}{2} - \nu\,, \quad \nu \equiv \sqrt{\frac{d^2}{4} + m^2}\ . \tag{4.5}$$

We have included a boundary term to the action (4.4), which is defined at $r = \epsilon$, the IR bulk regulator. To reproduce the double-trace deformation (4.2) we are instructed to use the boundary interaction

$$\mathcal{L}_{\text{boundary}} = \frac{1}{2} f\phi^2\ . \tag{4.6}$$

$f$ has a simple relation with the field theory $g$ in (4.2). A careful treatment gives

$$f = -\Delta - 2g\epsilon^{2\nu}\left(2\pi^{d/2}\frac{\Gamma(1-\nu)}{\Gamma(\Delta)}\right)\ . \tag{4.7}$$

Varying the action (4.4) with respect to $\phi$ we deduce the mixed Neumann/Dirichlet boundary conditions

$$f\ \phi(x,\epsilon) + \partial\phi(x,\epsilon)\cdot\hat{n} = 0\,, \tag{4.8}$$

where $\hat{n} = \epsilon\hat{r}$ is a unit vector specifying the normal to the boundary.

To compute correlation functions in the AdS/CFT correspondence, we evaluate the on-shell bulk action as a functional of the boundary source $\phi_b$. With mixed boundary conditions (4.8), the boundary value problem is

$$(\Box - m^2)\phi(x,r) = 0\,, \tag{4.9a}$$

$$f\ \phi(x,\epsilon) + \partial\phi(x,\epsilon)\cdot\hat{n} = \phi_b(x)\ . \tag{4.9b}$$

After the Fourier transformation

$$\phi(x_i,r) = \frac{1}{(2\pi)^{d/2}}\int d^d k\ e^{ik_i x_i}\phi(k_i,r) \tag{4.10}$$

the wave equation (4.9a) becomes (set $\phi(k,r) \equiv r^{d/2}\chi(k,r)$)

$$\left\{-\left(r\frac{d}{dr}\right)^2 + k^2 r^2 + \nu^2\right\}\chi(k,r) = 0\ . \tag{4.11}$$

The unique solution to (4.11), (4.9b), which is regular at $r \to \infty$, is given by the modified Bessel function of the 2nd kind $\mathcal{K}_\nu$. Consequently,

$$\phi(k,r) = \left( \frac{\phi_b(k)}{f\ \psi(k,\epsilon) + \partial\psi(k,\epsilon)\cdot\hat{n}} \right) \psi(k,r)\,, \quad \psi(k,r) \equiv r^{d/2}\mathcal{K}_\nu(kr)\ . \tag{4.12}$$

In particular, we notice that for any bare value of the double-trace coupling $g$, the solution is always the same Bessel function. What changes is the relation between the source $\phi_b$ and the asymptotic coefficients $\alpha, \beta$ in the near boundary ($r \to 0$) expansion of $\phi$

$$\phi(x,r) \sim r^{d-\Delta}\left[\alpha(x) + \mathcal{O}(r^2)\right] + r^{\Delta}\left[\beta(x) + \mathcal{O}(r^2)\right]\ . \tag{4.13}$$

Regularity, or equivalently the fact that we chose $\mathcal{K}_\nu$ as the solution of (4.11), has already fixed a linear relation between $\alpha$ and $\beta$.

Having said this, let us return to the matrix model case and the dual minimal strings. The analog of the wave equation (4.11) is the Wheeler-DeWitt (WdW) equation. For a minimal string with Liouville interaction

$$\delta\mathcal{S}_{\text{Liouville}} = \mu \int d^2z\ \mathcal{O}_{\text{matter}} e^{\alpha_+\varphi}\,, \quad \alpha_+ = b > 0 \tag{4.14}$$

the wavefunction $\Psi_\nu$ for a mode that corresponds to the vertex operator ($Q$ is the linear dilaton slope)

$$\mathcal{V}_\nu = \mathcal{O}_{\text{matter}} e^{-\frac{1}{2}(b\nu+Q)\varphi} = \Psi_\nu e^{-\frac{Q}{2}\varphi} \tag{4.15}$$

obeys, in the mini-superspace approximation, the WdW equation

$$\left\{ -\left(\ell\frac{d}{d\ell}\right)^2 + 4\mu\ell^2 + \nu^2 \right\} \Psi_\nu(\ell) = 0\ . \tag{4.16}$$

$\ell$ is related to the zero-mode of the Liouville coordinate $\varphi_0$ by the relation

$$\ell = e^{\frac{1}{2}b\varphi_0}\ . \tag{4.17}$$

The weak coupling end of Liouville theory lies at $\ell \to 0$, or equivalently $\varphi \to -\infty$.

The similarity between eqs. (4.11) and (4.16) is obvious. Again, the only solution that is regular in the "IR", *i.e.* the strong coupling region at $\ell \to \infty$, is

$$\Psi_\nu(\ell) \propto \mathcal{K}_\nu(2\sqrt{\mu}\ell)\ . \tag{4.18}$$

Writing $\mathcal{K}_\nu$ in terms of the modified Bessel functions of the 1st kind $\mathcal{I}_\nu$

$$\mathcal{K}_\nu = \frac{\pi}{2\sin(\pi\nu)}\left(\mathcal{I}_{-\nu} - \mathcal{I}_\nu\right) \tag{4.19}$$

we observe the weak coupling asymptotics

$$\mathcal{K}_\nu(\ell) \sim \frac{\pi}{2\sin(\pi\nu)}\left( \frac{2^\nu}{\Gamma(1-\nu)}\ell^\nu + \ldots - \frac{2^{-\nu}}{\Gamma(1+\nu)}\ell^{-\nu} + \ldots \right)\ . \tag{4.20}$$

Hence, regularity of the full wavefunction requires the presence of both the 'right' branch with $\nu < 0$, that satisfies the Seiberg bound [44], and the 'wrong' branch with $\nu > 0$, with a fixed reflection relation.

Deforming the matrix model with a double-trace operator, *e.g.* the operator $(\mathrm{Tr}\,\Phi^4)^2$, one finds two possible double-scaling limits: the standard one at double scaling parameter $g = 0$, and another one with different string susceptibility exponents at a special non-vanishing value of $g$ [23, 24]. These two critical points are analogous to the UV and IR fixed points of the CFT deformation in (4.2). The analysis of the string susceptibility exponents [23] and tree-level correlation functions [41] in the new double scaling limit indicates that one can reproduce the matrix model results in a Goulian-Li approach [45] to the minimal string by changing the Liouville dressing, from the right branch to the wrong branch, of the vertex operator that is dual to the matrix model operator involved in the double-trace deformation. Ref. [23] proposed this as a possible interpretation of the string theory dual to the deformed matrix model.

The statement about changing the branch of the Liouville dressing cannot mean that we modify the boundary conditions of the WdW equation in such a way that the new solution is solely $\mathcal{I}_{|\nu|}$. This solution is singular in the strong coupling region ($\ell \to \infty$), and one would need additional ingredients to explain this singularity, *e.g.* the singularity could be accounted for by the presence of a large number of ZZ branes [35] localized at the strong coupling region (see [46] for a related discussion). It is unclear, however, why ZZ branes would all of a sudden appear in the minimal string as we change $g$ continuously from the undeformed point to the new critical point. Also, this interpretation would not fit with our understanding of the analogous situations in the AdS/CFT correspondence, as described above.

Along the same lines, in recent years it has been understood that in Liouville theory both the standard Liouville interaction (4.14) and its dual

$$\widetilde{\delta\mathcal{S}}_{\text{Liouville}} = \tilde{\mu} \int d^2z \; \mathcal{O}_{\text{matter}} e^{\alpha_- \varphi} \,, \quad \alpha_- = -\left(Q + \frac{b}{2}\right) \tag{4.21}$$

have to be present with a fixed relation $\tilde{\mu} = \tilde{\mu}(\mu)$. Otherwise one cannot explain the structure of the exact two- and three-point functions [30 − 34].

In view of these observations, it is more appropriate to think of the deformed theory at tree-level in the following way. As a specific example, let us consider the case of the 2nd multi-critical matrix model deformed by $g(\mathrm{Tr}\,\Phi^4)^2$ at the new critical point where the partition function is given by (2.28). At tree-level, we can still think of the string theory dual of this matrix model as the usual (2,3) minimal string — the same as that at the undeformed $g = 0$ point. We should, however, modify the bulk/boundary dictionary and think of the amplitudes of the new theory as a function of the *dual* cosmological constant $\tilde{\mu}$. A way to rephrase this proposal is as follows. It is known that Liouville theory is symmetric under the simultaneous exchange of $\mu \leftrightarrow \tilde{\mu}$ and $\alpha_+ \leftrightarrow \alpha_-$ in (4.14), (4.21). Performing half of this transformation, *e.g.* $\mu \leftrightarrow \tilde{\mu}$ with $\alpha_+$ unchanged, gives a different theory. This is the theory we propose as the dual of the deformed matrix model at tree-level. This interpretation is consistent with our current knowledge of Liouville theory, the checks of the string susceptibility exponents and tree-level correlation functions in [23, 24, 41] and meshes nicely with the more general picture of double-trace deformations in the AdS/CFT correspondence.

We propose a similar interpretation of the double scaling limit (2.1.1) that leads to the modified partition function (2.27) of two coupled minimal strings. In that case, we re-interpret the original product partition sum as a function of the cosmological constant $t_- = Ct_1 - St_2$ and the dual $\tilde{t}_+ = S\tilde{t}_1 + C\tilde{t}_2$. Some evidence for this proposal is provided in appendix C where the sphere partition sum is analyzed from the continuum formalism viewpoint.

### 4.2.2 Beyond tree-level

The full partition function of the theories we considered in this paper is given by the Laplace transform of the original partition function. For instance, the free energy of the standard double scaling limit of the 2nd multi-critical matrix model is

$$F(t) = -\frac{2}{5}t^{5/2} - \frac{1}{24}\log t + \frac{7}{2160}t^{-5/2} + \mathcal{O}(t^{-5}) \ . \tag{4.22}$$

The Laplace transform of the exponential of this expression gives the free energy of the theory modified by $(\mathrm{Tr}\,\Phi^4)^2$ at the new critical point

$$F(\tilde{t}) = \frac{3}{5}\tilde{t}^{5/3} - \frac{7}{36}\log \tilde{t} + \frac{77}{960}\tilde{t}^{-5/3} + \mathcal{O}(\tilde{t}^{-10/3}) \ . \tag{4.23}$$

The relation between the cosmological constant $t$ and its dual $\tilde{t}$ is, up to a constant numerical factor,

$$t = \tilde{t}^{2/3} \ . \tag{4.24}$$

This relation reproduces the scaling of all terms in the perturbative expansion, and can even be used to reproduce exactly the tree-level result with appropriate fixing of the normalization of $t$, as explained in the previous subsection. It fails, however, to reproduce the higher loop coefficients. For example, it fails to reproduce the torus coefficient $-\frac{7}{36}$ giving instead the factor $-\frac{1}{36}$.

This implies that the tree-level interpretation of the previous subsection does not extend to higher loops, where one should think of the modified theory as a genuine NLST. It is tempting to speculate that a similar result holds in the higher dimensional AdS/CFT example, and that also there it is necessary to invoke the NLST structure to explain the properties of the modified string theory in the bulk.

## 4.3 Comments on the spacetime properties of the modified theories

The overall picture for the string theory dual of two double-trace deformed large $N$ matrix models at the critical point (2.1.1), (2.27) is a pair of minimal string theories with a subtle interaction induced on the level of the worldsheet via tiny wormholes. It is interesting to ask how such interactions manifest themselves in spacetime, or in other words how they enter in the string field theory action.

It is apparent from the scattering amplitudes computed in section 3 that there will be several types of contributions to the overall effective string field theory action, which will look schematically as

$$\mathcal{S}_{\text{total}} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_{\text{int}} \ . \tag{4.25}$$

$\mathcal{S}_1$ ($\mathcal{S}_2$) is the effective string field theory action of theory 1 (2). $\mathcal{S}_{\text{int}}$ is controlled by the double-trace angular parameter $\theta$ (see eq. (2.5)) and includes the interactions generated by the double-trace deformation. There are interactions that involve string fields of theory 1 or 2 separately, but also interactions coupling string fields of different theories (the non-vanishing correlation function $\langle\langle \mathcal{O}_1 \mathcal{O}_2 \rangle\rangle$ in section 3 is an example of such interactions). $\mathcal{S}_{\text{int}}$ is expected to be a non-local action. Aspects of spacetime non-locality in NLST in the context of the AdS/CFT correspondence have been discussed in [47, 48].

The higher dimensional generalization of this action in AdS/CFT will be a multi-gravity theory (when the gravity approximation is justified). $1/N$ effects will generate non-trivial interaction terms. One class of such interaction terms includes a potential for the gravitons, in particular a mass term for a linear combination of the gravitons. Such effective actions remind of the non-linear bi-gravity actions of [36].

# 5. Discussion

## 5.1 Summary and similarities with AdS/CFT examples

In this paper we found, by generalizing the analysis of [23, 24], that under certain conditions the product of $k$ large $N$ matrix models coupled together by a multi-trace operator admits new double scaling limits, which define holographically $k$ $c \leq 1$ non-critical string theories interacting with each other in a non-trivial manner. These cases provide one- and two-dimensional illustrations of the interacting (multi)-string theories proposed in [1, 2]. Specific examples of this phenomenon were given in the main text for $k = 2$ and in appendix B for $k = 3$. At least two conditions had to be met in order to obtain well-defined double scaling limits with a non-vanishing multi-trace coupling: ($a$) the single-trace operators that participate in the deformation should share the same scaling properties, and ($b$) we should fine-tune the multi-trace couplings to a special set of values. In the $k = 2$ examples of section 2 this special set comprised of a one-parameter family of double-trace couplings where the modified double scaling limit was giving rise to a modified partition function, which is related to the original product of two one-matrix model partition functions by a certain Laplace transform (see eq. (1.1)).

We argued that these deformed large-$N$ matrix models provide the holographic definition of an interacting product of $c \leq 1$ non-critical string theories. The interaction is mediated at the level of the worldsheet via non-local interactions, which induce tiny wormholes connecting worldsheets of the same or different string theories in a manner specified explicitly by the exact matrix model free energy. This non-local structure, which is evident in the correlation functions presented in section 3, is a special example of the non-local string theory construction anticipated on general grounds in [19] when multi-trace deformations are present in gauge/string dualities.

During the discussion of this structure and the possible interpretations of the modified matrix model as a modified string theory we had the chance to revisit an old claim of Klebanov [23], who proposed that a matrix model deformed by a critical double-trace deformation defines a string theory with a wrong branch tachyon condensate. We argued that one can make sense of this claim at tree-level as a change of the bulk/boundary dictionary, or equivalently as a re-interpretation of the standard minimal string free energy in terms of the dual cosmological constant. This observation is analogous to the tree-level interpretation of double-trace deformations in the AdS/CFT correspondence as mixed boundary conditions for the dual fields in AdS. We can check directly in the matrix model case that this picture cannot be extended beyond tree-level, where one should think of the modified string theories as *bona fide* NLSTs.

The matrix models in this paper and their dual string theories are interesting because they offer a set of clean, solvable examples of the general situation outlined in the beginning of the introduction: the holographic duality between a product of gauge theories deformed by a multi-trace deformation that preserves the product gauge group and string theory on a union of spaces. In that sense, the matrix model examples are useful precursors of analogous higher-dimensional cases in the AdS/CFT correspondence. The latter are interesting because they give us a rare chance to discuss the properties of massive, multi-graviton theories in a setting with a UV completion. The higher-dimensional AdS/CFT setups are the main focus of a companion paper [22].

Treating the matrix models as a playground for the corresponding AdS/CFT cases is further justified by the many similarities that they exhibit. We can summarize the basic parallels between the double-trace deformed matrix models and their higher dimensional CFT analogs with the following three points.

**Double-scaling limits vs fixed points of RG equations.** The double scaling limits in section 2 zoom around points in parameter space where the matrix models exhibit a critical behavior. At special values of the double-trace parameters, *e.g.* when equation (2.25) holds, a new critical behavior arises, but as we move away from this special submanifold in coupling space either we recover the critical behavior of the original undeformed theory, or the theory enters into a branched polymer phase where no sensible critical behavior exists. As a necessary condition for the existence of the new critical behavior, the scaling properties of the single-trace operators participating in the deformation have to be the same.

A similar picture of critical double-trace deformations arises in higher dimensional QFTs [22]. Let us consider a general double-trace deformation of a $d$-dimensional CFT of the form

$$\delta\mathcal{S} = \int d^d x \, \Big[ g_{11}(\mathcal{O}_1)^2 + g_{22}(\mathcal{O}_2)^2 + 2g_{12}\mathcal{O}_1\mathcal{O}_2 \Big] \,, \tag{5.1}$$

where $\mathcal{O}_1$, $\mathcal{O}_2$ are single-trace operators in two theories (1 and 2 respectively) with scaling dimensions $\Delta_1, \Delta_2$. The deformation (5.1) is the direct analog of the matrix model deformation (2.3). By analyzing the one-loop $\beta$-functions of the double-trace couplings in conformal perturbation theory at leading order in $1/N$ one finds new fixed points away from the origin when the scaling dimensions are either equal or are related by the equation $\Delta_1 = d - \Delta_2$. We encountered a similar condition on the scaling properties of the deforming operators of the matrix models in the previous paragraph. When new fixed points exist, there is a one-parameter family of them, which is the analog of the one-parameter family of critical behaviors captured by the modified free energy (1.1). Away from the fixed points the RG flow drives us back towards the undeformed theory at the origin, or away from the origin towards a region where conformal perturbation theory breaks down. In the first case, the double-trace deformation acts as an irrelevant operator around the origin — this is the analog of recovering the standard double scaling limit in the matrix model. The second case is more like the branched polymer phase in the matrix model.

Beyond tree-level, the RG equations in the higher dimensional cases receive $1/N$ corrections which displace the critical circle and introduce a coupling between single-trace, double-trace and other multi-trace couplings. At the same time, $1/N$ corrections in the bulk produce an effective potential for the gravitons. In particular, they lift the mass of a linear combination of the gravitons and make the dual multi-gravity theory massive. The structure of the RG equations on the boundary predicts that the space will remain a union of AdS spaces only if we fine-tune the double-trace couplings to a special set of values. Otherwise, loop effects will backreact to the background. In those cases, where we can trust the perturbation theory and the RG flow on the boundary drives the theory back towards the undeformed point, we expect the dual gravity description of the IR physics to be captured by a decoupled union of AdS spaces. In this way, the theory develops dynamically a region of spacetime where the mass of the gravitons is washed away. A more detailed discussion of these issues can be found in [22].

**The bulk/boundary correspondence at tree-level.** Another similarity between the matrix model and the higher dimensional CFT cases concerns the tree-level interpretation of the deformation on the dual string theory or gravity side. In both cases the partition function of the deformed theory at the new critical points is related to that of the original theory by a Legendre transform. In the bulk, we still have a product of standard minimal string theories or a product of supergravity theories on a union of AdS spaces, however, the bulk/boundary dictionary is now modified. In the matrix model case, we re-interpret the standard amplitudes of minimal strings as a function of the dual cosmological constant. In the AdS case, we put mixed boundary conditions on the dual fields.

**The NLST structure.** Finally, in both cases and to all orders in the $1/N$ expansion, the theory in the bulk is an NLST along the lines of [19]. In the matrix model case, we have an expression exact to all orders in perturbation theory for the modified free energy, which is given by a Laplace transform. This allows for an explicit derivation of the NLST rules in this case. In the higher dimensional AdS/CFT cases, we do not have the luxury of such exact expressions but a qualitatively similar structure is anticipated.

### 5.2 Open problems and possible extensions

We would like to close with a short list of open problems. An important issue is the non-perturbative stability of the modified string theories in this paper and generalizations thereof. The minimal $(p,q)$ bosonic string theories have well-known problems at the non-perturbative level, hence Laplace transforms of the form (2.27), (2.29) are at best good expressions in a perturbative expansion. It would be nice to re-examine the effects of multi-trace deformations in examples which are well-defined non-perturbatively. Perhaps, the unitary matrix models of [40] are a good starting point for such an exercise.

In this paper we discussed exclusively what happens to closed strings when we couple two or more non-critical string theories together. It would be interesting to explore the effect of this coupling also to open string sectors. For instance, FZZT branes [49, 50] are represented in the matrix model by the microscopic loop operator $\log\det(x-\Phi)$. It would be interesting to analyze how the multi-trace deformations affect amplitudes that involve FZZT branes, *i.e.* amplitudes that involve a number of insertions of the determinant operator $\det(x-\Phi)$, and what this implies for the worldsheet theory.

Another interesting problem is to analyze the pattern of interactions in the case of a large number of string theories coupled by non-local interactions. A simple example along these lines is to consider $M$ $c=0$ matrix models coupled pairwise with double-trace interactions in a circular fashion. This is reminiscent of the matrix model couplings that define the $c=1$ matrix theory in terms of $c=0$ matrix models, however, the difference here is that the nearest-neighbor couplings are of double-trace type. Such constructions should involve a limit where $N \gg M \gg 1$.

One of the main motivations behind this work is cosmology and the potential for interesting time-dependent solutions in multi-gravity theories. One could attempt to analyze such solutions in two-dimensional toy examples extending the ideas in [51] to systems of coupled $c=1$ string theories.

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## A. Products of matrix quantum mechanics models

In this appendix we discuss double scaling limits in a double-trace deformation of a product of two matrix quantum mechanics theories. Double scaling limits in double-trace deformed matrix quantum mechanics theories were discussed originally in [52, 53].

For us the partition function of interest is[1]

$$\mathcal{Z} = \int D\Phi_1(t) D\Phi_2(t)\; e^{-N\int_0^{2\pi R} dt[\mathrm{Tr}(\frac{1}{2}\dot\Phi_1^2+\frac{1}{2}\Phi_1^2-\lambda_1\Phi_1^3)+\mathrm{Tr}(\frac{1}{2}\dot\Phi_2^2+\frac{1}{2}\Phi_2^2-\lambda_2\Phi_2^3)]} \times \\ e^{-\int_0^{2\pi R} dt[g_{11}(\mathrm{Tr}\,\Phi_1^3)^2+2g_{12}\,\mathrm{Tr}\,\Phi_1^3\,\mathrm{Tr}\,\Phi_2^3+g_{22}(\mathrm{Tr}\,\Phi_2^3)^2]}\;. \qquad (A.1)$$

[1]For simplicity, we will set the ranks of the two matrix models $N_1$ and $N_2$ to be equal. This will not affect the final results in any significant way.

The matrix model lives on a compact one-dimensional spacetime with radius $R$. The standard double scaling limit of this model at $g_{11} = g_{22} = g_{12} = 0$ describes holographically the decoupled product of two $c = 1$ non-critical string theories.

As in the zero-dimensional matrix model case of section 2, it is convenient to re-exress the double-trace deformation as

$$g_{11}(\operatorname{Tr}\Phi_1^3)^2 + g_{22}(\operatorname{Tr}\Phi_2^3)^2 + 2g_{12}\operatorname{Tr}\Phi_1^3\operatorname{Tr}\Phi_2^3 = \\ = r_1\left(C\operatorname{Tr}\Phi_1^3 + S\operatorname{Tr}\Phi_2^3\right)^2 + r_2\left(-S\operatorname{Tr}\Phi_1^3 + C\operatorname{Tr}\Phi_2^3\right)^2 , \tag{A.2}$$

where again we will use the notation $C \equiv \cos\theta$ and $S \equiv \sin\theta$. The same relations between $g_{ij}$ and $r_1, r_2$ and $\theta$ hold as in subsection 2.1.

Following [24], it will be useful to perform the following Fourier transforms

$$P_i = \int_0^{2\pi R} dt\ \operatorname{Tr}\Phi_i^3 , \qquad C_{i,n} = \sqrt{2}\int_0^{2\pi R} dt\ \cos\frac{nt}{R}\operatorname{Tr}\Phi_i^3 , \\ S_{i,n} = \sqrt{2}\int_0^{2\pi R} dt\ \sin\frac{nt}{R}\operatorname{Tr}\Phi_i^3 , \qquad i = 1,2 . \tag{A.3}$$

Then we can recast the double-trace interactions as an infinite sum of squares

$$\int_0^{2\pi R} dt\left(C\operatorname{Tr}\Phi_1^3 + S\operatorname{Tr}\Phi_2^3\right)^2 = \\ = \frac{1}{2\pi R}\left\{(CP_1 + SP_2)^2 + \sum_{n=1}^{\infty}\left[(CC_{1,n} + SC_{2,n})^2 + (CS_{1,n} + SS_{2,n})^2\right]\right\} , \tag{A.4a}$$

$$\int_0^{2\pi R} dt\left(-S\operatorname{Tr}\Phi_1^3 + C\operatorname{Tr}\Phi_2^3\right)^2 = \\ = \frac{1}{2\pi R}\left\{(-SP_1 + CP_2)^2 + \sum_{n=1}^{\infty}\left[(-SC_{1,n} + CC_{2,n})^2 + (-SS_{1,n} + CS_{2,n})^2\right]\right\} \tag{A.4b}$$

and proceed following the strategy of section 2.

First, we distinguish between two cases: $\det g = r_1 r_2 \neq 0$ and $\det g = r_1 r_2 = 0$. The former case is a simpler version of the latter, hence this is the one we will discuss mostly. Assuming $\det g \neq 0$, we can use repeatedly the identity (2.10) to recast the partition function (A.1) into the form

$$\mathcal{Z} = \int_{-\infty}^{\infty} dy_0 dx_0 \prod_{n=1}^{\infty} dy_n dx_n dz_n dw_n\ e^{\frac{\pi R N^2}{2}\left[\frac{y_0^2+y_n^2+z_n^2}{r} + \frac{x_0^2+x_n^2+w_n^2}{\rho}\right]} \times \\ \int D\Phi_1(t) D\Phi_2(t)\ e^{-N\int_0^{2\pi R} dt[\operatorname{Tr}(\frac{1}{2}\dot\Phi_1^2 + \frac{1}{2}\Phi_1^2 - (\lambda_1 + Cy_0 - Sx_0)\Phi_1^3) + \operatorname{Tr}(\frac{1}{2}\dot\Phi_2^2 + \frac{1}{2}\Phi_2^2 - (\lambda_2 + Sy_0 + Cx_0)\Phi_2^3)]} \\ e^{N\sum_{n=1}^{\infty}[(Cy_n - Sx_n)C_{1,n} + (Sy_n + Cx_n)C_{2,n} + (Cz_n - Sw_n)S_{1,n} + (Sz_n + Cw_n)S_{2,n}]} . \tag{A.5}$$

In order to proceed further we need additional information about the partition sum of a single $c = 1$ MQM model. The free energy of this theory is

$$\log\int D\Phi(t) e^{-N\int_0^{2\pi R} dt\ \left[\operatorname{Tr}(\frac{1}{2}\dot\Phi^2 + \frac{1}{2}\Phi^2 - (\lambda + y_0)\Phi^3\right] - \sum_{n=1}^{\infty}(y_n C_n + z_n S_n)} = \\ = 2\pi R N^2\left(-a_1 x + \frac{1}{2}a_2 x^2\right) + F_0(x, N^2) + F_1(x, y_n, z_n, N^2) , \tag{A.6a}$$

where

$$F_0(x,N^2) \,=\, RN^2\left(\frac{1}{2}a_3x^2/\log x+\cdots\right)-\frac{1}{24}\left(R+\frac{1}{R}\right)\log x+\cdots\,, \quad \text{(A.6b)}$$

$$F_1(x,y_n,z_n,N^2) \,=\, \pi RN^2\sum_{n=1}^{\infty}\left(y_n^2+z_n^2\right)\left(b_n+b'_n\,(x/\log x)^{n/R}+\cdots\right)+\cdots\,, \quad \text{(A.6c)}$$

$$x \,=\, c_2-\lambda-y_0\;. \quad \text{(A.6d)}$$

In our case, we have to deal with the decoupled product of two such theories, so we set

$$y=c_2-(\lambda_1+Cy_0-Sx_0)\,,\quad x=c_2-(\lambda_2+Sy_0+Cx_0)\,, \quad \text{(A.7)}$$

which can be inverted to

$$y_0=\Delta_1-Cy-Sx\,,\quad x_0=\Delta_2+Sy-Cx\;. \quad \text{(A.8)}$$

In these expressions $\Delta_1$ and $\Delta_2$ are still defined as in (2.15a), (2.15b), (2.16).

Plugging this information back into (A.5) we obtain

$$\mathcal{Z} \,=\, \int_{-\infty}^{\infty}dxdy\prod_{n=1}^{\infty}dy_ndx_ndz_ndw_n\; e^{\frac{\pi RN^2}{2}\left[\frac{y_n^2+z_n^2}{r}+\frac{x_n^2+w_n^2}{\rho}+\frac{(\Delta_1-Cy-Sx)^2}{r}+\frac{(\Delta_2+Sy-Cx)^2}{\rho}\right]}\times$$
$$e^{2\pi RN^2(-a_1x+\frac{1}{2}a_2x^2-a_1y+\frac{1}{2}a_2y^2)+F_0(x,N^2)+F_0(y,N^2)}\times$$
$$e^{F_1(y,Cy_n-Sx_n,Cz_n-Sw_n,N^2)+F_1(x,Sy_n+Cx_n,Sz_n+Cw_n,N^2)}\;. \quad \text{(A.9)}$$

The next step is to diagonalize the quadratic terms in the auxiliary parameters and depending on the sign of eigenvalues take a suitable double scaling limit.

First, let us consider the quadratic part associated with the auxiliary parameters $x, y$

$$\mathcal{S}_2(x,y)=\frac{\pi RN^2}{2\det g}\left[(2a_2\det g+g_{11})x^2+(2a_2\det g+g_{22})y^2-2g_{12}xy\right]\;. \quad \text{(A.10)}$$

To diagonalize this expression, we rotate to the variables $x_\pm$ defined by the linear transformation

$$y=U_y^+x_++U_y^-x_-\,,\quad x=U_x^+x_++U_x^-x_- \quad \text{(A.11)}$$

with $U_y^\pm$ and $U_x^\pm$ the same as $U_1^\pm$ and $U_2^\pm$ in eq. (2.19). Then we get

$$\mathcal{S}_2(x,y)=-N^2(M_+^2x_+^2+M_-^2x_-^2) \quad \text{(A.12)}$$

with

$$M_\pm^2=2\pi Rm_\pm^2 \quad \text{(A.13)}$$

and $m_\pm^2$ as in eq. (2.21b).

The corresponding masses for the higher auxiliary modes $y_n, z_n, x_n, w_n$ are

$$z_n,\;y_n\;:\quad m_{zy}^2\,=\,-\frac{\pi R}{2}\left(\frac{1}{r}+b_n\right)\,, \quad \text{(A.14a)}$$

$$x_n,\;w_n\;:\quad m_{xw}^2\,=\,-\frac{\pi R}{2}\left(\frac{1}{\rho}+b_n\right)\;. \quad \text{(A.14b)}$$

Since $b_n < a_2$ for all $n \geq 1$ [24], we deduce the inequalities

$$m_{zy}^2 > -\frac{\pi R}{2}\left(\frac{1}{r} + 2a_2\right) , \quad m_{xw}^2 > -\frac{\pi R}{2}\left(\frac{1}{\rho} + 2a_2\right) , \tag{A.15}$$

which, say assuming $r_1 > r_2$, imply

$$m_{zy}^2 > M_+^2 , \quad m_{xw}^2 > M_-^2 . \tag{A.16}$$

Consequently, the auxiliary parameters $z_n, y_n, x_n, w_n$ are massive and can be integrated out as long as $M_\pm^2 \geq 0$. In that case, the partition function (A.9) takes the simpler form

$$\mathcal{Z} = \int_{-\infty}^{\infty} dx_+ dx_- e^{\pi R N^2 (E^+ x_+ + E^- x_-)} e^{-N^2 (m_+^2 x_+^2 + m_-^2 x_-^2)} \times \\ e^{F_0(U_x^+ x_+ + U_x^- x_-, N^2) + F_0(U_y^+ x_+ + U_y^- x_-, N^2)} , \tag{A.17a}$$

where

$$E^\pm = \left(-2a_1 - \frac{C\Delta_1}{r} + \frac{S\Delta_2}{\rho}\right) U_y^\pm - \left(2a_1 + \frac{S\Delta_1}{r} + \frac{C\Delta_2}{\rho}\right) U_x^\pm . \tag{A.17b}$$

If both $m_\pm^2$ are positive, we can further integrate out both $x_\pm$ to obtain the direct product of two $c = 1$ partition functions

$$\mathcal{Z} = e^{F_0\left(U_x^+ \frac{E^+ \pi R}{2m_+^2} + U_x^- \frac{E^- \pi R}{2m_-^2}, N^2\right) + F_0\left(U_y^+ \frac{E^+ \pi R}{2m_+^2} + U_y^- \frac{E^- \pi R}{2m_-^2}, N^2\right)} . \tag{A.18}$$

If one of the masses squared is zero, $m_+^2 = 0$, and the other positive, we can integrate out $x_-$ to obtain a Legendre transform formula analogous to eq. (2.27)

$$\mathcal{Z} = \int_{-\infty}^{\infty} dx_+ \; e^{\pi R N^2 E^+ x_+} e^{F_0\left(U_x^+ x_+ + U_x^- \frac{E^- \pi R}{2m_-^2}, N^2\right) + F_0\left(U_y^+ x_+ + U_y^- \frac{E^- \pi R}{2m_-^2}, N^2\right)} . \tag{A.19}$$

Finally, when both $m_\pm = 0$, $g_{12} = 0$, $g_{11} = g_{22} = r_1 = r_2 = -\frac{1}{2a_2}$. In this case, we are dealing with a decoupled product of two $c = 1$ string theories that have been deformed individually by double-trace deformations to their critical points as in [24].

## B. Triple intersections

In the main text we mostly restricted our attention to pairs of matrix models or CFTs coupled by a double-trace deformation. In this appendix we consider a triple intersection: the product of three theories coupled by a triple-trace operator. As a concrete example, we will analyze the case of three 2nd multicritical matrix models. The partition function of the theory of interest is[2]

$$\mathcal{Z} = \int \prod_{i=1}^{3} D\Phi_i e^{-N \sum_{i=1}^{3} \left[\frac{1}{2}\Phi_i^2 + \lambda_i \Phi_i^4\right] - \frac{1}{N} \sum_{i,j,k=1}^{3} g_{ijk} (\mathrm{Tr}\, \Phi_i^4)(\mathrm{Tr}\, \Phi_j^4)(\mathrm{Tr}\, \Phi_k^4)} . \tag{B.1}$$

[2]Again for simplicity, we take the ranks of all three matrices to be equal.

The coefficient of the triple-trace interaction has been scaled as $N^{-1}$, so that the whole interaction term scales as $N^2$. By definition, the parameters $g_{ijk}$ are fully symmetric in $i, j, k = 1, 2, 3$.

A general multi-trace deformation that involves $n$ single-trace operators can always be recast in terms of single-trace interactions by introducing $2n$ auxiliary field integrations [54]. In the case of the triple-trace deformation (B.1) this amounts to introducing three auxiliary pairs of parameters $(\sigma_i, v_i)$ and writing

$$\begin{aligned}
\mathcal{Z} &= \int \prod_{i=1}^{3} D\Phi_i d\sigma_i dv_i \; e^{-N\sum_{i=1}^{3}\left[\frac{1}{2}\Phi_i^2+\lambda_i\Phi_i^4\right]-\frac{1}{N}\sum_{i,j,k=1}^{3} g_{ijk}\sigma_i\sigma_j\sigma_k+\sum_{i=1}^{3} v_i(\sigma_i-\mathrm{Tr}\,\Phi_i^4)} \\
&= \prod_{i=1}^{3}\int d\sigma_i dx_i \; e^{\sum_{i=1}^{3}\left[N^2(-a_1x_i+\frac{1}{2}a_2x_i^2)+F(x_i,N^2)+N(x_i-c_2-\lambda_i)\sigma_i\right]-\frac{1}{N}\sum_{i,j,k}^{3} g_{ijk}\sigma_i\sigma_j\sigma_k} \\
&\equiv \prod_{i=1}^{3}\int d\sigma_i dx_i \; e^{\mathcal{F}(x_i,\sigma_i,N^2)} \; . \qquad \text{(B.2)}
\end{aligned}$$

In the first equality the $v_i$ integrations are defined by analytic continuation. In the second equality we changed integration variables from $v_i$ to $x_i$

$$v_i = N(x_i - c_2 - \lambda_i) \; . \qquad \text{(B.3)}$$

The introduction of the auxiliary parameters $(\sigma_i, v_i)$ generalizes the trick of eq. (2.10) to an arbitrary multi-trace interaction.

**Saddle point expansion.** The saddle point equations for the integral expression (B.2) are

$$x_i \quad : \quad -a_1 + a_2\bar{x}_i + \frac{1}{N^2}\frac{\partial F}{\partial x_i} + \frac{\bar{\sigma}_i}{N} = 0\,, \qquad \text{(B.4a)}$$

$$\sigma_i \quad : \quad N(\bar{x}_i - c_2 - \lambda_i) - \frac{3}{N}\left[g_{iii}\bar{\sigma}_i^2 + 2\sum_{k\neq i} g_{iik}\bar{\sigma}_i\bar{\sigma}_k + \sum_{j,k\neq i} g_{ijk}\bar{\sigma}_j\bar{\sigma}_k\right] = 0 \; . \qquad \text{(B.4b)}$$

We have denoted the saddle point values of $(\sigma_i, x_i)$ as $(\bar{\sigma}_i, \bar{x}_i)$.

The expansion of the free energy $\mathcal{F}(x_i, \sigma_i, N^2)$ around the saddle point values involves the second derivatives

$$\left.\frac{\partial^2\mathcal{F}}{\partial x_i\partial x_j}\right|_{\bar{x},\bar{\sigma}} = \delta_{ij}\left(N^2 a_2 + \left.\frac{\partial^2 F}{\partial x_i^2}\right|_{\bar{x}}\right)\,, \qquad \text{(B.5a)}$$

$$\left.\frac{\partial^2\mathcal{F}}{\partial \sigma_i\partial \sigma_j}\right|_{\bar{x},\bar{\sigma}} = -\frac{6}{N}\left[\delta_{ij}g_{iii}\bar{\sigma}_i + \left(g_{iij}\bar{\sigma}_i + g_{ijj}\bar{\sigma}_j + \sum_{k\neq i,k\neq j} g_{ijk}\bar{\sigma}_k\right)_{i\neq j}\right]\,, \qquad \text{(B.5b)}$$

$$\left.\frac{\partial^2\mathcal{F}}{\partial x_i\partial \sigma_j}\right|_{\bar{x},\bar{\sigma}} = N\delta_{ij} \; . \qquad \text{(B.5c)}$$

Several simplifications are possible in the large $N$ limit. First, anticipating the double scaling limit, where $\bar{x}_i \to 0$, we can drop the $\frac{\partial^2 F}{\partial x_i^2}\Big|_{x_i}$ terms in (B.5a) as subleading. Then, the leading terms in the saddle point equation (B.4a) give

$$\bar{\sigma}_i = N a_1 \tag{B.6}$$

and eq. (B.5b) becomes

$$\frac{\partial^2 \mathcal{F}}{\partial \sigma_i \partial \sigma_j}\Bigg|_{\bar{x},\bar{\sigma}} = -6a_1 \left[ \delta_{ij} g_{iii} + \left( g_{iij} + g_{ijj} + \sum_{k\neq i, k\neq j} g_{ijk} \right)_{i\neq j} \right] \equiv -6a_1 f_{ij} \ . \tag{B.7}$$

With these simplifications the partition function (B.2) becomes after shifting $x_i \to x_i - \frac{\sigma_i}{N a_2}$

$$\mathcal{Z} = \prod_{i=1}^{3} \int dx_i d\sigma_i \ e^{\mathcal{F}(\bar{x},\bar{\sigma}_i,N^2)+\sum_{i=1}^{3}\frac{a_2}{2}N^2 x_i^2 - \frac{1}{2}\sum_{i,j=1}^{3}\mathcal{M}_{ij}^2 \sigma_i \sigma_j + \mathcal{O}(x^3,\sigma^3)} \ , \tag{B.8}$$

where we define the auxiliary field mass matrix $\mathcal{M}_{ij}$ as

$$\mathcal{M}_{ij}^2 = \frac{\delta_{ij}}{a_2} + 6a_1 f_{ij} \ . \tag{B.9}$$

The properties of $\mathcal{Z}$ depend crucially on whether the matrix $\mathcal{M}^2$ is positive definite or not.

As an illuminating special case, we will consider in detail what happens when the only non-vanishing triple-trace parameter $g_{ijk}$ is $g_{123} \equiv g$. The eigenvalues of the mass squared matrix

$$\mathcal{M}^2 = \begin{pmatrix} \frac{1}{a_2} & 6a_1 g & 6a_1 g \\ 6a_1 g & \frac{1}{a_2} & 6a_1 g \\ 6a_1 g & 6a_1 g & \frac{1}{a_2} \end{pmatrix} \tag{B.10}$$

are

$$\lambda_- = \frac{1 - 6a_1 a_2 g}{a_2} \ \ (\text{double eigenvalue}) \, , \quad \lambda_+ = \frac{1 + 12 a_1 a_2 g}{a_2} \ . \tag{B.11}$$

$\mathcal{M}^2$ is positive definite inside the interval

$$g_-^* < g < g_+^* \, , \quad g_-^* = -\frac{1}{12 a_1 a_2} \, , \ g_+^* = \frac{1}{6 a_1 a_2} \ . \tag{B.12}$$

At the lower end of this interval, $g = g_-^*$, one of the $\sigma$ eigenvectors becomes massless. At the upper end, $g = g_+^*$, two of the $\sigma$ eigenvectors become massless. The critical values $g_\pm^*$ are therefore good candidates for the definition of a new set of double scaling limits.

**Double scaling limits.** We proceed to analyze the exact partition function

$$\begin{aligned} \mathcal{Z} &= \prod_{i=1}^{3} \int d\sigma_i dx_i \ e^{\sum_{i=1}^{3}\left[N^2(-a_1 x_i + \frac{1}{2}a_2 x_i^2) + F(x_i,N^2) + N(x_i + 6g a_1^2 - \Delta_i)\sigma_i\right] - \frac{6}{N} g \sigma_1 \sigma_2 \sigma_3} \\ &\equiv \prod_{i=1}^{3} \int d\sigma_i dx_i \ e^{\mathcal{F}(x_i,\sigma_i,N^2)} \ . \end{aligned} \tag{B.13}$$

in a double scaling limit where

$$N \to \infty\,, \quad \Delta_i \equiv c_2 + \lambda_i + 6ga_1^2 \to 0 \tag{B.14}$$

with a particular combination, to be specified, kept fixed.

It will be convenient to begin with the following manipulations:

(*i*) Shift the auxiliary parameters $\sigma_i \to \sigma_i + N(a_1 - a_2 x_i)$.

(*ii*) Diagonalize the $3\times 3$ mass matrix $\mathcal{M}^2$ (see eq. (B.10)) with the linear transformation[3]

$$x_i = \sum_j \mathcal{U}_i^j X_j\ . \tag{B.15}$$

In the new variables the action $\mathcal{F}$ becomes

$$\begin{aligned}\mathcal{F}(\sigma_i, X_i, N^2) = & \sum_{i=1}^{3}\Big[F(\mathcal{U}_i^j X_j, N) - N\Delta_i\sigma_i + N^2 a_2\Delta_i\mathcal{U}_i^j X_j - Na_2\sigma_i\mathcal{U}_i^j X_j\Big] - \\ & -\frac{1}{2}N^2 a_2^2\sum_{i,j}\Big[\lambda_-(X_1^2 + X_2^2) + \lambda_+ X_3^2\Big] \\ & -6ga_1\Big[\sigma_1\sigma_2 - Na_2(\sigma_1\mathcal{U}_2^j X_j + \sigma_2\mathcal{U}_1^j X_j) + (\text{cyclic})\Big] - \\ & -\frac{6g}{N}\Big(\sigma_1 - Na_2\mathcal{U}_1^j X_j\Big)\Big(\sigma_2 - Na_2\mathcal{U}_2^j X_j\Big)\Big(\sigma_3 - Na_2\mathcal{U}_3^j X_j\Big)\ .\end{aligned} \tag{B.16}$$

With minor re-arranging, we recast (B.16) as

$$\begin{aligned}\mathcal{F}(\sigma_i, X_i, N^2) = & \sum_{i=1}^{3} F(\mathcal{U}_i^j X_j, N) - N\Delta_i\sigma_i - \frac{1}{2}N^2 a_2^2\left[\lambda_-(X_1^2 + X_2^2) + \lambda_+ X_3^2\right] + G(\sigma_i) \\ & + S^j(\sigma) X_j + R^{jk}(\sigma) X_j X_k + R^{ijk} X_i X_j X_k\,,\end{aligned} \tag{B.17}$$

where

$$\begin{aligned} G(\sigma) &= -N\Delta_i\sigma_i - 6ga_1(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3) - \frac{6g}{N}\sigma_1\sigma_2\sigma_3\,, && \text{(B.18a)}\\ S^j(\sigma) &= N^2 a_2\Delta_i\mathcal{U}_i^j - Na_2\sigma_i\mathcal{U}_i^j + 6ga_1a_2N\Big[(\mathcal{U}_2^j + \mathcal{U}_3^j)\sigma_1 + (\mathcal{U}_1^j + \mathcal{U}_3^j)\sigma_2 + (\mathcal{U}_1^j + \mathcal{U}_2^j)\sigma_3\Big] \\ &\quad + 6ga_2\Big[\mathcal{U}_3^j\sigma_1\sigma_2 + \mathcal{U}_2^j\sigma_1\sigma_3 + \mathcal{U}_1^j\sigma_2\sigma_3\Big]\,, && \text{(B.18b)}\\ R^{jk}(\sigma) &= -6gNa_2^2\Big[\mathcal{U}_1^j\mathcal{U}_2^k\sigma_3 + \mathcal{U}_2^j\mathcal{U}_3^k\sigma_1 + \mathcal{U}_1^j\mathcal{U}_3^k\sigma_2\Big]\,, && \text{(B.18c)}\\ R^{ijk} &= 6gN^2a_2^3\,\mathcal{U}_1^i\mathcal{U}_2^j\mathcal{U}_3^k\ . && \text{(B.18d)}\end{aligned}$$

We will consider three distinct cases: (*a*) both $\lambda_-$, $\lambda_+ > 0$, (*b*) $\lambda_- = 0$, $\lambda_+ > 0$ or (*c*) $\lambda_- > 0$, $\lambda_+ = 0$. When one of the lambdas becomes negative there will be no double-scaling limit, so we will not discuss this situation separately.

[3]The coefficients $\mathcal{U}_i^j$ are functions of $a_1, a_2, g$ whose explicit form we will not determine here.

The first case, where both lambdas are positive, occurs when $g$ lies inside the interval (B.12). To cancel the linear term $X_j S^j$ in (B.17) we shift the $X$ variables setting

$$X_i = Z_i + \tilde{S}_i\,, \tag{B.19}$$

where by definition

$$\tilde{S}_1 = \frac{S^1}{N^2 a_2^2 \lambda_-}\,, \quad \tilde{S}_2 = \frac{S^2}{N^2 a_2^2 \lambda_-}\,, \quad \tilde{S}_3 = \frac{S^3}{N^2 a_2^2 \lambda_+}\,. \tag{B.20}$$

Then we scale the parameters appearing in the free energy in the following way

$$Z_i = N^{-4/5} z_i\,, \quad \tilde{S}_i = N^{-4/5} s_i\,. \tag{B.21}$$

The latter follows from the scaling

$$\Delta_i = \delta_i N^{-4/5}\,, \quad \sigma_i = N^{1/5} u_i\,. \tag{B.22}$$

(B.21) allows us to cancel the explicit dependence of the single-matrix model free energies $F(\mathcal{U}_i^j Z_j + \mathcal{U}_i^j \tilde{S}_j, N^2)$ on $N$ and to re-express them as functions $F(\mathcal{U}_i^j z_j + \mathcal{U}_i^j s_j)$ of $z_j$, $s_j$ only. With this scaling the quadratic and cubic terms $R^{jk} X_j X_k$, $R^{ijk} X_i X_j X_k$ become subleading and can be ignored. We are left with the single-matrix free energies $F$, quadratic terms in $z_i$ and linear and quadratic terms in $s_i$. After some trivial algebra, we obtain the following expression for the partition function

$$\mathcal{Z} = \prod_{i=1}^{3} \int dz_i ds_i\; e^{\sum_{i=1}^{3} F(\mathcal{U}_i^j z_j + \tilde{\mathcal{U}}_i^j s_j + t_i)} e^{-\frac{N^{2/5} a_2^2}{2}\left[\lambda_-(z_1^2+z_2^2)+\lambda_+ z_3^2\right]} e^{-N^{2/5} d^i s_i^2}\,, \tag{B.23}$$

where the new coefficients $\tilde{\mathcal{U}}_j^i$, $d^i$ are functions of $a_1, a_2, g$ and $t_i$ are functions of $a_1, a_2, g$ and $\delta_i$ — the scaling parameters that were defined in (B.22). The last two Gaussian terms in (B.23) become delta functions in the large $N$ limit,[4] hence we end up with the partition function of three decoupled one-matrix models

$$\mathcal{Z}(t_1, t_2, t_3) = \prod_{i=1}^{3} e^{F(t_i)}\,. \tag{B.24}$$

Now we come to the more interesting case where one of the lambda eigenvalues vanishes. Let us consider first the case with $\lambda_- = 0$, $\lambda_+ = \frac{1+12a_1a_2g}{a_2} = \frac{3}{a_2} > 0$. Eq. (B.17) simplifies a bit to become

$$\begin{aligned}\mathcal{F}(\sigma_i, X_i, N^2) = &\sum_{i=1}^{3} F(\mathcal{U}_i^j X_j, N) - N\Delta_i\sigma_i - \frac{1}{2}N^2 a_2^2 \lambda_+ X_3^2 + G(\sigma_i) \\ &+ S^j(\sigma) X_j + R^{jk}(\sigma) X_j X_k + R^{ijk} X_i X_j X_k\,.\end{aligned} \tag{B.25}$$

This time we shift the $X$ variables by setting

$$X_1 = Z_1\,, \quad X_2 = Z_2\,, \quad X_3 = Z_3 + \frac{S^3}{N^2 a_2^2 \lambda_+} = Z_3 + \tilde{S}_3 \tag{B.26}$$

[4] We define the large $N$ limit of the last Gaussian factor by analytic continuation when $d^i$ are negative.

and then we scale in the following way

$$Z_i = N^{-4/5} z_i \,, \quad \tilde{S}_3 = N^{-4/5} s_3 \,, \quad \sigma_i = N^{1/5} u_i \ . \tag{B.27}$$

The function $G(\sigma)$ breaks up in this limit into two pieces scaling in different ways

$$G(\sigma) = N^{2/5} \tilde{G}(u) - N^{6/5} \Delta_i u_i \,, \tag{B.28}$$

where

$$\tilde{G}(u) = -6 g a_1 (u_1 u_2 + u_2 u_3 + u_1 u_3) \ . \tag{B.29}$$

The last two terms in (B.25) are again subleading in the large $N$ limit and can be ignored. Hence, we are left at this stage with the action

$$\begin{aligned}\mathcal{F}(u_i, z_i, N^2) = & \sum_{i=1}^{3} F(\mathcal{U}_i^j z_j + \mathcal{U}_i^3 s_3) + N^{2/5} \tilde{G}(u) - N^{6/5} \Delta_i u_i \\ & + N^{-4/5}(z_1 S^1 + z_2 S^2) - \frac{1}{2} N^{2/5} a_2^2 \lambda_+ z_3^2 + \frac{1}{2} N^{2/5} a_2^2 \lambda_+ s_3^2 \ . \end{aligned} \tag{B.30}$$

The $z_3^2$ term in (B.30) will contribute a delta function and the $z_3$ integral will localize at $z_3 = 0$. This simplifies the action $\mathcal{F}$ a bit further to

$$\begin{aligned}\mathcal{F}(u_i, z_1, z_2, N^2) = & \sum_{i=1}^{3} F(\mathcal{U}_i^1 z_1 + \mathcal{U}_i^2 z_2 + \mathcal{U}_i^3 s_3) + N^{2/5} \tilde{G}(u) - N^{6/5} \Delta_i u_i \\ & + N^{-4/5}(z_1 S^1 + z_2 S^2) + \frac{1}{2} N^{2/5} a_2^2 \lambda_+ s_3^2 \ . \end{aligned} \tag{B.31}$$

The term $N^{-4/5}(z_1 S^1 + z_2 S^2)$ has a $u_i$-independent piece

$$N^{6/5} a_2 \Delta_i (\mathcal{U}_i^1 z_1 + \mathcal{U}_i^2 z_2) \tag{B.32}$$

which we will require to stay finite in the large $N$ limit. This can be achieved with the scaling

$$\mathcal{U}_i^\alpha \Delta_i = N^{-6/5} \delta^\alpha \,, \quad \alpha = 1, 2 \ . \tag{B.33}$$

The remaining combination of $\Delta_i$'s will be scaled in the standard way

$$\mathcal{U}_i^3 \Delta_i = N^{-4/5} \delta \ . \tag{B.34}$$

Hence, after some algebra $\mathcal{F}$ reduces to an expression of the form

$$\begin{aligned}\mathcal{F}(u_i, z_1, z_2, N^2) = & \sum_{i=1}^{3} F\Big(\mathcal{U}_i^1 z_1 + \mathcal{U}_i^2 z_2 + \mathcal{U}_i^3 \frac{\delta}{a_2 \lambda_+} + \mathcal{U}_i^3 \mathcal{N}^j u_j\Big) + a_2 \delta_i (\mathcal{U}_i^1 z_1 + \mathcal{U}_i^2 z_2) \\ & + N^{2/5} \left[ u_i (\mathcal{K}^{1i} z_1 + \mathcal{K}^{2i} z_2) + \mathcal{L}^{ij} u_i u_j \right] + \mathcal{P}^i u_i \,, \end{aligned} \tag{B.35}$$

where the constants $\mathcal{N}^j, \mathcal{K}^{\alpha i}, \mathcal{L}^{ij}, \mathcal{P}^i$ (for $i, j = 1, 2, 3$, $\alpha = 1, 2$) depend only $a_1, a_2$.

To obtain the final result, we diagonalize the quadratic term $\mathcal{L}^{ij}u_iu_j$ and use the large $N$ limit to localize the $u_i$ integrals at $u_i = 0$. This kills all the contributions which are linear and homogeneous in $u_i$. Then, defining the new scaling parameters

$$\tilde{t}^1 = a_2\delta_i\mathcal{U}_i^1\ ,\quad \tilde{t}^2 = a_2\delta_i\mathcal{U}_i^2\ ,\quad t_3 = \frac{\delta}{a_2\lambda_+} \tag{B.36}$$

and renaming for aesthetic reasons $z_1$, $z_2$ as $t_1, t_2$ we get the double-scaled partition function

$$\mathcal{Z}(\tilde{t}_1, \tilde{t}_2, t_3) = \int_{-\infty}^{\infty} dt_1 dt_2\ e^{\tilde{t}^1 t_1 + \tilde{t}^2 t_2 + \sum_{i=1}^3 F(\mathcal{U}_i^1 t_1 + \mathcal{U}_i^2 t_2 + \mathcal{U}_i^3 t_3)}\ . \tag{B.37}$$

This expression, which generalizes eq. (2.27), is a double Laplace transform of the original partition function. A double integration is natural, since it is a double eigenvalue (*i.e.* $\lambda_-$) that we tune to zero.

Similar manipulations can be performed in the last case of interest: $\lambda_- = \frac{3}{2a_2} > 0$, $\lambda_+ = 0$. We will skip the gory details and present the final, double-scaled expression for the partition function

$$\mathcal{Z}(\tilde{t}_1, t_2, t_3) = \int_{-\infty}^{\infty} dt_1\ e^{\tilde{t}^1 t_1 + \sum_{i=1}^3 F(\mathcal{U}_i^1 t_1 + \mathcal{U}_i^2 t_2 + \mathcal{U}_i^3 t_3)}\ . \tag{B.38}$$

## C. String susceptibility exponents in the continuum approach

The double scaling limit (2.1.1) and the resulting partition function (2.27) give the holographic definition of two interacting minimal (2,3) bosonic string theories. In this appendix, we present evidence for the tree-level interpretation of these theories in subsection 4.2.1 by studying the scaling properties of the deformed sphere partition function using the continuum formalism of the minimal strings.

For completeness, and in order to set the notation, let us begin by recalling a few well-known facts about the one-matrix model case. For the main example in this appendix, the double scaling limit of the partition function of the 2nd multicritical matrix model

$$\mathcal{Z} = \int D\Phi\ e^{-N\,\mathrm{Tr}\left[\frac{1}{2}\Phi^2 + \lambda\Phi^4\right]} \tag{C.1}$$

one finds the free energy

$$F(t) = -\frac{2}{5}t^{5/2} - \frac{1}{24}\log t + \frac{7}{2160}t^{-5/2} + \mathcal{O}(t^{-10}) \tag{C.2}$$

where $t$ is the double scaling parameter.

When we add a double-trace deformation to obtain the partition function

$$\mathcal{Z} = \int D\Phi\ e^{-N\,\mathrm{Tr}\left[\frac{1}{2}\Phi^2 + \lambda\Phi^4\right] - g(\mathrm{Tr}\,\Phi^4)^2} \tag{C.3}$$

and tune $g$ to the critical value $-\frac{1}{2a_2}$, a new double-scaling limit is possible giving the free energy

$$F(\tilde{t}) = \frac{3}{5}\tilde{t}^{5/3} - \frac{7}{36}\log\tilde{t} + \frac{77}{960}\tilde{t}^{-5/3} + \cdots\ . \tag{C.4}$$

This expression can be deduced from (C.2) and the Laplace transform

$$F(\tilde{t}) = \log \int_{-\infty}^{\infty} dt\ e^{t\tilde{t}+F(t)}\ . \tag{C.5}$$

The string susceptibility exponents $\gamma$ are defined at any genus $g$ in terms of the exponents of the scaling parameters at each order in the expansion of the free energy

$$F(t) = \ldots + \#\ t^{(2-\gamma)\chi/2} + \ldots\ , \quad \chi = 2 - 2g\ . \tag{C.6}$$

The above double scaling limits exhibit different string susceptibility exponents: (C.2) gives $\gamma = -\frac{1}{2}$, whereas (C.4) gives $\gamma = \frac{1}{3}$.

There is a simple way to determine the critical exponents $\gamma$ from the continuum formulation of the dual string theory. For $(p, q)$ minimal models coupled to gravity the Liouville interaction takes the form

$$\delta \mathcal{S}_{\text{Liouville}} = \mu \int d^2 z\ \mathcal{O}_{\text{min}}\ e^{\alpha_+ \phi}\ , \quad \alpha_+ = -\frac{p+q-1}{\sqrt{2pq}} \tag{C.7}$$

where $\mathcal{O}_{\text{min}}$ is the matter primary field with the lowest dimension

$$h_{\text{min}} = \frac{1-(p-q)^2}{4pq}\ . \tag{C.8}$$

The Liouville path integral in the partition function can be computed by separating the zero mode $\phi_0$ and performing the relevant integral

$$\int_{-\infty}^{\infty} d\phi_0\ e^{Q\chi\phi_0/2 - \mu\ e^{\alpha_+\phi_0}} = \frac{1}{\alpha_+}\Gamma\left(\frac{Q\chi}{2\alpha_+}\right)\mu^{-\frac{Q\chi}{2\alpha_+}}\ . \tag{C.9}$$

In this expression $Q = \sqrt{2}\frac{p+q}{\sqrt{pq}}$ is the linear dilaton slope. Identifying the Liouville interaction constant $\mu$ with the matrix model scaling parameter $t$ we deduce the critical exponent

$$\gamma = 2 + \frac{Q}{\alpha_+}\ . \tag{C.10}$$

For example, when $(p, q) = (2, 3)$ (corresponding to the 2nd multicritical matrix model) we have $Q = \frac{5}{\sqrt{3}}$ and $\alpha_+ = -\frac{2}{\sqrt{3}}$ and therefore

$$\gamma = -\frac{2}{p+q-1} = -\frac{1}{2} \tag{C.11}$$

reproducing the above matrix model result.

For the other double scaling limit (C.4), which involves a critical double-trace deformation, it has been proposed by ref. [23] that one should consider a (2,3) minimal string with the wrong branch tachyon in (C.7), *i.e.* one should replace $\alpha_+ \to \alpha_- = -\frac{p+q+1}{\sqrt{2pq}}$. Indeed, for $(p, q) = (2, 3)$ this substitution reproduces the matrix model result $\gamma = 1/3$. In subsection 4.2.1 we rephrased this proposal as a transformation from the cosmological constant $\mu$ to the dual cosmological constant $\tilde{\mu}$.

We now proceed to apply a similar logic to the two-matrix model case (2.27). The sphere contribution to the free energy

$$F(\tilde{t}_+, t_-) = \log \int_{-\infty}^{\infty} dt_+ \; e^{\tilde{t}_+ t_+ + F_1(U_1^+ t_+ + U_1^- t_-) + F_2(U_2^+ t_+ + U_2^- t_-)} \tag{C.12}$$

is the leading contribution in the saddle point approximation. The saddle point value of $t_+$ is given implicitly by the following equation

$$\tilde{t}_+ = U_1^+ (U_1^+ t_+ + U_1^- t_-)^{3/2} + U_2^+ (U_2^+ t_+ + U_2^- t_-)^{3/2} \; . \tag{C.13}$$

We should solve this equation for $t_+$ in terms of $\tilde{t}_+, t_-$ and then insert the result into the saddle point expression for $F$

$$F(\tilde{t}_+, t_-) = \tilde{t}_+ t_+ - \frac{2}{5}(U_1^+ t_+ + U_1^- t_-)^{5/2} - \frac{2}{5}(U_2^+ t_+ + U_2^- t_-)^{5/2} + \cdots \; . \tag{C.14}$$

It seems difficult to obtain a closed expression for generic $t_-$, but one can easily deduce an expression that involves a perturbative expansion in $t_-$. Indeed, one can show that the sphere contribution to the free energy admits an expansion of the form

$$F(\tilde{t}_+, t_-) = \sum_{n=0}^{\infty} f_n(\theta) \; \tilde{t}_+^{\frac{5-2n}{3}} t_-^n + \cdots \, , \tag{C.15}$$

where $f_n(\theta)$ are functions of $\theta$ that can be determined. In an effort to reproduce this expansion from Liouville theory, we now turn to the continuum formalism of the minimal string.

Before adding the double-trace deformation, the total free energy $F(t_1, t_2)$ is simply the sum of the free energies of the two independent constituent matrix models and the sphere contribution has the following form in terms of Liouville zero mode integrals

$$F(t_1, t_2)\big|_{\text{sphere}} \sim \int_{-\infty}^{\infty} d\phi_1 e^{Q\phi_1 - t_1 e^{\alpha_+ \phi_1}} + \int_{-\infty}^{\infty} d\phi_2 e^{Q\phi_2 - t_2 e^{\alpha_+ \phi_2}} \sim t_1^{-\frac{Q}{\alpha_+}} + t_2^{-\frac{Q}{\alpha_+}} = t_1^{5/2} + t_2^{5/2} \, . \tag{C.16}$$

Rotating to a new mixed basis of coupling constants

$$t_1 = U_1^+ t_+ + U_1^- t_- \, , \quad t_2 = U_2^+ t_+ + U_2^- t_- \tag{C.17}$$

we may rewrite the same expression as

$$F(t_+, t_-)\big|_{\text{sphere}} \sim \int_{-\infty}^{\infty} d\phi_1 e^{Q\phi_1 - (U_1^+ t_+ + U_1^- t_-) e^{\alpha_+ \phi_1}} + \int_{-\infty}^{\infty} d\phi_2 e^{Q\phi_2 - (U_2^+ t_+ + U_2^- t_-) e^{\alpha_+ \phi_2}} \, . \tag{C.18}$$

To obtain the worldsheet version of the matrix model result (C.12), (C.15) we transform from $t_+ e^{\alpha_+ \phi_{1,2}}$ to the dual cosmological constant interactions $\tilde{t}_+ e^{\alpha_- \phi_{1,2}}$ leaving the $t_-$ terms invariant. This implies the Liouville interaction

$$\delta \mathcal{S}_{\text{total}} = \int d^2 z_1 \left( U_1^+ \tilde{t}_+ e^{\alpha_- \phi_1} + U_1^- t_- e^{\alpha_+ \phi_1} \right) + \int d^2 z_2 \left( U_2^+ \tilde{t}_+ e^{\alpha_- \phi_2} + U_2^- t_- e^{\alpha_+ \phi_2} \right) \, . \tag{C.19}$$

The sphere free energy is then

$$F(\tilde{t}_+, t_-) \sim \int_{-\infty}^{\infty} d\phi_1 e^{Q\phi_1 - U_1^+ \tilde{t}_+ e^{\alpha_- \phi_1} - U_1^- t_- e^{\alpha_+ \phi_1}} + \int_{-\infty}^{\infty} d\phi_2 e^{Q\phi_2 - U_2^+ \tilde{t}_+ e^{\alpha_- \phi_2} - U_2^- t_- e^{\alpha_+ \phi_2}} . \tag{C.20}$$

Expanding the exponentials in powers of $t_-$ and then evaluating the Liouville zero mode integrals term by term we obtain

$$F(\tilde{t}_+, t_-)\big|_{\text{sphere}} \sim \sum_{n=0}^{\infty} h_n(\theta)\, \tilde{t}_+^{-\frac{Q+n\alpha_+}{\alpha_-}}\, t_-^n = \sum_{n=0}^{\infty} h_n(\theta)\, \tilde{t}_+^{\frac{5-2n}{3}}\, t_-^n \tag{C.21}$$

reproducing the matrix model expansion (C.15). We have not attempted to make a precise matching of the functions $f_n(\theta), h_n(\theta)$. This would involve going beyond the zero mode integrals. We regard the expansion (C.21) as evidence in favor of the picture proposed in subsection 4.2.1.

## References


[1] E. Kiritsis, *Product CFTs, gravitational cloning, massive gravitons and the space of gravitational duals*, *JHEP* **11** (2006) 049 [hep-th/0608088].

[2] O. Aharony, A.B. Clark and A. Karch, *The CFT/AdS correspondence, massive gravitons and a connectivity index conjecture*, *Phys. Rev.* **D 74** (2006) 086006 [hep-th/0608089].

[3] M. Porrati, *Higgs phenomenon for 4D gravity in Anti de Sitter space*, *JHEP* **04** (2002) 058 [hep-th/0112166]; *Higgs phenomenon for the graviton in AdS space*, *Mod. Phys. Lett.* **A 18** (2003) 1793 [hep-th/0306253].

[4] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, *Proc. Roy. Soc. Lond.* **A 173** (1939) 211.

[5] C. Aragone and S. Deser, *Constraints on gravitationally coupled tensor fields*, *Nuovo Cim.* **3A** (1971) 709; *Consistency problems of hypergravity*, *Phys. Lett.* **B 86** (1979) 161;
D.G. Boulware and S. Deser, *Inconsistency of finite range gravitation*, *Phys. Lett.* **B 40** (1972) 227;
S. Deser and A. Waldron, *Inconsistencies of massive charged gravitating higher spins*, *Nucl. Phys.* **B 631** (2002) 369 [hep-th/0112182].

[6] A.I. Vainshtein, *To the problem of nonvanishing gravitation mass*, *Phys. Lett.* **B 39** (1972) 393.

[7] N. Arkani-Hamed, H. Georgi and M.D. Schwartz, *Effective field theory for massive gravitons and gravity in theory space*, *Ann. Phys. (NY)* **305** (2003) 96 [hep-th/0210184].

[8] C. Deffayet and J.-W. Rombouts, *Ghosts, strong coupling and accidental symmetries in massive gravity*, *Phys. Rev.* **D 72** (2005) 044003 [gr-qc/0505134];
P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, *Ghosts in massive gravity*, *JHEP* **09** (2005) 003 [hep-th/0505147];
G. Gabadadze and A. Gruzinov, *Graviton mass or cosmological constant?*, *Phys. Rev.* **D 72** (2005) 124007 [hep-th/0312074].

[9] C. Bachas and P.M.S. Petropoulos, *Topological models on the lattice and a remark on string theory cloning*, *Commun. Math. Phys.* **152** (1993) 191 [hep-th/9205031].

[10] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, *Inconsistency of interacting, multigraviton theories*, *Nucl. Phys.* **B 597** (2001) 127 [hep-th/0007220].

[11] V.A. Rubakov and P.G. Tinyakov, *Infrared-modified gravities and massive gravitons*, arXiv:0802.4379.

[12] G.R. Dvali, G. Gabadadze and M. Porrati, *4D gravity on a brane in 5D Minkowski space*, *Phys. Lett.* **B 485** (2000) 208 [hep-th/0005016].

[13] E. Kiritsis, N. Tetradis and T.N. Tomaras, *Thick branes and 4D gravity*, *JHEP* **08** (2001) 012 [hep-th/0106050]; *Induced gravity on RS branes*, *JHEP* **03** (2002) 019 [hep-th/0202037].

[14] S.L. Dubovsky and V.A. Rubakov, *On models of gauge field localization on a brane*, *Int. J. Mod. Phys.* **A 16** (2001) 4331 [hep-th/0105243].

[15] M.A. Luty, M. Porrati and R. Rattazzi, *Strong interactions and stability in the DGP model*, *JHEP* **09** (2003) 029 [hep-th/0303116].

[16] V.A. Rubakov, *Strong coupling in brane-induced gravity in five dimensions*, hep-th/0303125.

[17] R. Gregory, *The three burials of Melquiades DGP*, *Prog. Theor. Phys. Suppl.* **172** (2008) 71 [arXiv:0801.1603].

[18] T. Damour, I.I. Kogan and A. Papazoglou, *Non-linear bigravity and cosmic acceleration*, *Phys. Rev.* **D 66** (2002) 104025 [hep-th/0206044].

[19] O. Aharony, M. Berkooz and E. Silverstein, *Multiple-trace operators and non-local string theories*, *JHEP* **08** (2001) 006 [hep-th/0105309]; *Non-local string theories on $AdS_3 \times S^3$ and stable non-supersymmetric backgrounds*, *Phys. Rev.* **D 65** (2002) 106007 [hep-th/0112178].

[20] I.R. Klebanov, *String theory in two-dimensions*, hep-th/9108019.

[21] P.H. Ginsparg and G.W. Moore, *Lectures on $2-D$ gravity and $2-D$ string theory*, hep-th/9304011.

[22] E. Kiritsis and V. Niarchos, *A holographic perspective on the instabilities of massive graviton theories*, to appear.

[23] I.R. Klebanov, *Touching random surfaces and Liouville gravity*, *Phys. Rev.* **D 51** (1995) 1836 [hep-th/9407167].

[24] I.R. Klebanov and A. Hashimoto, *Nonperturbative solution of matrix models modified by trace squared terms*, *Nucl. Phys.* **B 434** (1995) 264 [hep-th/9409064].

[25] E. Witten, *Multi-trace operators, boundary conditions and AdS/CFT correspondence*, hep-th/0112258.

[26] M. Berkooz, A. Sever and A. Shomer, *Double-trace deformations, boundary conditions and spacetime singularities*, *JHEP* **05** (2002) 034 [hep-th/0112264].

[27] W. Mueck, *An improved correspondence formula for AdS/CFT with multi-trace operators*, *Phys. Lett.* **B 531** (2002) 301 [hep-th/0201100].

[28] P. Minces, *Multi-trace operators and the generalized AdS/CFT prescription*, *Phys. Rev.* **D 68** (2003) 024027 [hep-th/0201172].

[29] A.C. Petkou, *Boundary multi-trace deformations and OPEs in AdS/CFT correspondence*, *JHEP* **06** (2002) 009 [hep-th/0201258].

[30] H. Dorn and H.J. Otto, *On correlation functions for noncritical strings with $c \leq 1, d \geq 1$*, *Phys. Lett.* **B 291** (1992) 39 [hep-th/9206053].

[31] H. Dorn and H.J. Otto, *Two and three point functions in Liouville theory*, *Nucl. Phys.* **B 429** (1994) 375 [hep-th/9403141].

[32] A.B. Zamolodchikov and A.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys.* **B 477** (1996) 577 [hep-th/9506136].

[33] J. Teschner, *Liouville theory revisited*, *Class. and Quant. Grav.* **18** (2001) R153 [hep-th/0104158].

[34] J. Teschner, *A lecture on the Liouville vertex operators*, *Int. J. Mod. Phys.* **A19S2** (2004) 436 [hep-th/0303150].

[35] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[36] T. Damour and I.I. Kogan, *Effective Lagrangians and universality classes of nonlinear bigravity*, *Phys. Rev.* **D 66** (2002) 104024 [hep-th/0206042].

[37] S.R. Das, A. Dhar, A.M. Sengupta and S.R. Wadia, *New critical behavior in $D = 0$ large-$N$ matrix models*, *Mod. Phys. Lett.* **A 5** (1990) 1041.

[38] D.J. Gross and A.A. Migdal, *Nonperturbative two-dimensional quantum gravity*, *Phys. Rev. Lett.* **64** (1990) 127;
M.R. Douglas and S.H. Shenker, *Strings in less than one-dimension*, *Nucl. Phys.* **B 335** (1990) 635;
E. Brézin and V.A. Kazakov, *Exactly solvable field theories of closed strings*, *Phys. Lett.* **B 236** (1990) 144.

[39] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar diagrams*, *Commun. Math. Phys.* **59** (1978) 35.

[40] I.R. Klebanov, J.M. Maldacena and N. Seiberg, *Unitary and complex matrix models as $1-D$ type 0 strings*, *Commun. Math. Phys.* **252** (2004) 275 [hep-th/0309168].

[41] J.L.F. Barbon, K. Demeterfi, I.R. Klebanov and C. Schmidhuber, *Correlation functions in matrix models modified by wormhole terms*, *Nucl. Phys.* **B 440** (1995) 189 [hep-th/9501058].

[42] S.S. Gubser and I.R. Klebanov, *A universal result on central charges in the presence of double-trace deformations*, *Nucl. Phys.* **B 656** (2003) 23 [hep-th/0212138].

[43] T. Hartman and L. Rastelli, *Double-trace deformations, mixed boundary conditions and functional determinants in AdS/CFT*, *JHEP* **01** (2008) 019 [hep-th/0602106].

[44] N. Seiberg, *Notes on quantum Liouville theory and quantum gravity*, *Prog. Theor. Phys. Suppl.* **102** (1990) 319.

[45] M. Goulian and M. Li, *Correlation functions in Liouville theory*, *Phys. Rev. Lett.* **66** (1991) 2051.

[46] D. Kutasov, K. Okuyama, J.-w. Park, N. Seiberg and D. Shih, *Annulus amplitudes and ZZ branes in minimal string theory*, *JHEP* **08** (2004) 026 [hep-th/0406030].

[47] O. Aharony, M. Berkooz and E. Silverstein, *Non-local string theories on* $AdS_3 \times S^3$ *and stable non-supersymmetric backgrounds*, *Phys. Rev.* **D 65** (2002) 106007 [hep-th/0112178].

[48] O. Aharony, M. Berkooz and B. Katz, *Non-local effects of multi-trace deformations in the AdS/CFT correspondence*, *JHEP* **10** (2005) 097 [hep-th/0504177].

[49] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory. I: boundary state and boundary two-point function*, hep-th/0001012.

[50] J. Teschner, *Remarks on Liouville theory with boundary*, hep-th/0009138.

[51] J.L. Karczmarek and A. Strominger, *Matrix cosmology*, *JHEP* **04** (2004) 055 [hep-th/0309138].

[52] F. Sugino and O. Tsuchiya, *Critical behavior in* $c = 1$ *matrix model with branching interactions*, *Mod. Phys. Lett.* **A 9** (1994) 3149 [hep-th/9403089].

[53] S.S. Gubser and I.R. Klebanov, *A modified* $c = 1$ *matrix model with new critical behavior*, *Phys. Lett.* **B 340** (1994) 35 [hep-th/9407014].

[54] J.L.F. Barbon and C. Hoyos-Badajoz, *AdS/CFT, multitrace deformations and new instabilities of nonlocal string theories*, *JHEP* **01** (2004) 049 [hep-th/0311274].